Differential Decay

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(Dated: March 22, 2024)

I. ROTATION OF POLARIZATION VECTOR IN MULTIPLE SCATTERING

The cumulative scattering angle is due to electrostatic forces. In passage through electric fields, at the magic momentum, the orientation of the spin with respect to the particle trajectory is 'frozen'. The components of the polarization parallel and perpendicular to the particle momentum are preserved. The scattering angle is necessarily accompanied by a change in the direction of the polarization so that $\mathbf{s}\cdot\mathbf{p}$ is conserved.

A. Coordinates

In the bmad environment the momentum vector is

$$\begin{aligned} p_x &= p_0(\text{coord\%vec(2)}) \\ p_y &= p_0(\text{coord\%vec(4)}) \\ p_z &= \sqrt{p - p_x^2 - p_z^2} = \left(p_0^2(\text{coord\%vec(6)}+1)^2 - p_x^2 - p_y^2\right)^{1/2} \end{aligned}$$

where p_0 is the reference momentum and $p = \sqrt{p_z^2 + p_x^2 + p_y^2}$ and $(\text{coord\%vec(6)}) = \frac{p-p_0}{p_0}$. The polarization vector is $\mathbf{s} = (s_x, s_y, s_z) = (\text{coord\%spin(1)}, \text{coord\%spin(2)}, \text{coord\%spin(3)})$.

Initial and final momenta and polarization are $\mathbf{p}_i, \mathbf{s}_i$ and $\mathbf{p}_f, \mathbf{s}_f$. For convenience transform to a coordinate system where $\mathbf{p}_i = p_i\hat{x}$, and initial and final momenta lie in the x-z plane by a series of rotations.

- First rotate about the z-axis by an angle θ so that the initial momentum is in the x-y plane

$$R_z(\theta)\begin{pmatrix} p_x^i \\ p_y^i \\ p_z^i \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p_x^i \\ p_y^i \\ p_z^i \end{pmatrix} = \begin{pmatrix} p'_x \\ 0 \\ p'_z \end{pmatrix} \tag{1}$$

  Evidently $\theta = \tan^{-1}\frac{p_y^i}{p_x^i}$.

- Next rotate about the y'-axis by an angle ϕ to a coordinate system where the incoming momentum is exclusively along the z''-axis.

$$R_{y'}(\phi)\begin{pmatrix} p'_x \\ 0 \\ p'_z \end{pmatrix} = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}\begin{pmatrix} p'_x \\ 0 \\ p'_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p''_z \end{pmatrix} \tag{2}$$

  where $\phi = \tan^{-1}\frac{p'_x}{p'_z}$.

  In the new coordinate system the initial polarization and final momentum become

$$\begin{aligned} \mathbf{s}''_i &= R'_y(\phi)R_z(\theta)\mathbf{s}_i \\ \mathbf{p}''_f &= R'_y(\phi)R_z(\theta)\mathbf{p}_f \end{aligned}$$

- Finally rotate once again by an angle α about the z'' axis so that the final momentum lies in the x-z plane.

$$R_{z''}(\alpha)\mathbf{p}''_f = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p''_{fx} \\ p''_{fy} \\ p''_{fz} \end{pmatrix} = \begin{pmatrix} p'''_{fx} \\ 0 \\ p'''_{fz} \end{pmatrix} \tag{3}$$

  $\alpha = \tan^{-1}\frac{p''_{fx}}{p''_{fz}}$. In the triple primed coordinate system, the scattering is confined to the $x''' - z'''$ plane. The scattering angle, $\theta_{scatter} = \tan^{-1}\frac{p'''_{fx}}{p'''_{fz}}$.

- Transform the intial polarization to the triple primed system.

$$\mathbf{s}_i''' == R_{z''}(\alpha)\mathbf{s}_i''$$

The effect of the electric field is to rotate the component of polarization in the scattering plane by the scattering angle. The final polarization in the triple primed coordinate system is

$$\mathbf{s}_f''' = R_{z'''}(\theta_{scatter})\mathbf{s}_i''' = \begin{pmatrix} \cos\theta_s & \sin\theta_s & 0 \\ -\sin\theta_s & \cos\theta_s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_{ix}''' \\ s_{iy}''' \\ s_{iz}''' \end{pmatrix} = R_z(\theta_{scatter})R_z(\alpha)R_y(\phi)R_z(\theta)\mathbf{s}_i \tag{4}$$

The last step is to transform back into the lab frame.

$$\mathbf{s}_f = R_z^{-1}(\theta)R_y^{-1}(\phi)R_z^{-1}(\alpha)\mathbf{s}_f''' \tag{5}$$