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Betatron Oscillations and E-field Systematic

Because of the finite curvature of the quadrupoles, the E-field correction to ω_a can depend on the betatron amplitude as well as the momentum. The muon that circulates at the magic radius with zero betatron oscillation amplitude will see zero electric field and no correction is required. But the muon with that same momentum and finite betatron amplitude will oscillate about the magic radius and its precession will indeed be effected by the E-field. If the quad field were strictly antisymmetric about the magic radius and the betatron trajectory strictly symmetric, the net contribution would be zero. But neither condition is true.

Electrostatic quadrupole nonlinearity

Consider the electric field in the quadrupoles. Laplace's equation in two dimensions and cartesian coordinates is

$$\nabla^2 V(x,y) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1}$$

The potential corresponding to a pure quadrupole field is one (of many) solutions.

$$V = \frac{1}{2}k(x^2 - y^2)$$

for some constant k. The electric field is

$$E = \vec{\nabla} V = k(x\hat{\imath} - y\hat{\jmath})$$

and of course the divergence is zero. The electric field is linear in both x and y. Higher order terms may appear (and indeed in the g-2 quads they well appear due to the geometry of the plates), but the symmetry (in the 2-D cartesion limit permits only those terms that are odd in x, y. In particular there is no sextupole ($\sim kx^2$) dependence that would be symmetric in displacement.

In the limit of finite curvature (as in the g-2 geometry) it is more appropriate to represent the fields in cylindrical coordinates, where Laplace's equation is

$$\nabla^2 V(\rho, z, \phi) = \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right) V = 0$$

We assume the quads are continuous so that there is no dependence on the angular coordinate ϕ. Then the simplest possible solution is

$$V(\rho, z) = k\left(\frac{1}{2}(\frac{\rho^2}{\rho_0^2} - 1) - \ln\frac{\rho}{\rho_0} - \left(\frac{z}{\rho_0}\right)^2\right)$$

and the electric field

$$\vec{\nabla} V = \frac{1}{2}k\left((\rho - \frac{\rho_0^2}{\rho})\hat{\rho} - 2z\hat{z}\right) \tag{2}$$

If we write $\rho = \rho_0 + x$ where ρ_0 is the magic radius, then

$$\vec{E} \sim k\left((x - \frac{x^2}{2\rho_0} + \ldots)\hat{\rho} - z\hat{z}\right) \tag{3}$$

Evidently, Maxwell's equations require a term quadratic in displacement. The quadratic term is equivalent to a sextupole component that will effect the chromaticity, complicate the correction of the E-field systematic, and possibly drive a third order resonance.

The above solution is not unique. One can write alternative solutions where the quadratic (x^2) term does not appear in the radial field, but then it always turns up as ((xz) in the vertical direction. There exist no solutions without some sextupole component. A fit to the Wanwei Wu field map would properly identify this term.

Note that the sextupole component that inevitably goes along for the ride with a linear component, due to the curvature of the quad plates, cannot not drive the $3\nu_y = 1$ resonance that appears in the quad scans. That would require a skew sextupole that is forbidden by the quad symmetry, and would depend on some misalignment.

In any event, there is a symmetric component of the E-field which will contribute to the E-field systematic.

Path length

The radial electric field along the trajectory of a muon with equilibrium radial offset x_e and betatron amplitude x_β is

$$
\begin{aligned}
E_r(s) &= k(x_e + x_\beta) - \frac{1}{2\rho_0}(x_e + x_\beta)^2 + \ldots \\
E_r(s) &= k(\eta\delta + x_\beta) - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2 + \ldots
\end{aligned}
$$

where $x_e = \eta\delta = \eta\frac{\Delta p}{p}$ and we assume the 'simple' solution to Laplace's equation discussed above. The muon trajectory is along the path dl as shown in Figure 1. The path s is the reference orbit. The

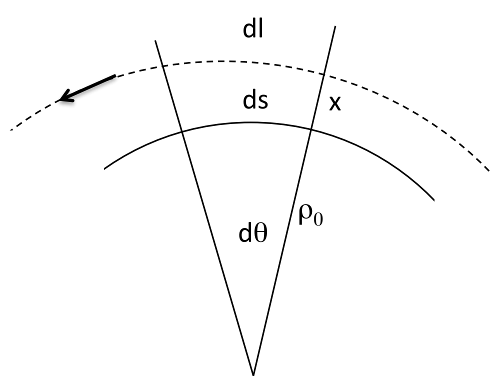

Figure 1: Curvilinear coordinate system. The integrated path length for the part of the trajectory at $\rho > \rho_0$ is greater than the length of the path for $\rho < \rho_0$.

average electric field along the trajectory is

$$\langle E_r(s)\rangle = k\langle\Big(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2\Big)\rangle \tag{4}$$

$$= \frac{k}{L}\int_0 \Big(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2\Big)\, dl \tag{5}$$

$$= \frac{k}{L}\int_0 \Big(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2\Big)\,(1 + x_\beta/\rho_0)ds \tag{6}$$

here in that last step we use $d\phi = \frac{ds}{\rho_0} = \frac{dl}{(\rho_0+x)}$, and L is the length of the trajectory. Using $x_\beta = x_{\beta 0}\cos\phi(s)$, $\langle E_r\rangle$ becomes

$$\begin{aligned}
\langle E_r(s)\rangle &= k\int \Big(\eta\delta + x_{\beta0}\cos\phi - \frac{1}{2\rho_0}(\eta\delta + x_{\beta0}\cos\phi)^2\Big)\, d\phi(1 + x/\rho_0) \\
&= k\int \Big(\eta\delta + x_{\beta0}\cos\phi + \frac{1}{2\rho_0}(\eta\delta + x_{\beta0}\cos\phi)^2\Big)\,(1 + x_{\beta0}/\rho_0\cos\phi + \eta\delta/\rho_0)d\phi \\
&= k\left(\eta\delta - \frac{1}{2\rho_0}((\eta\delta)^2 + \frac{1}{2}x_{\beta0}^2) + \frac{1}{2\rho_0}x_{\beta0}^2 + (\eta\delta)^2 + \ldots\right) \\
&= k\left(\eta\delta + \frac{1}{2\rho_0}((\eta\delta)^2 + \frac{1}{2}x_{\beta0}^2) + \ldots\right)
\end{aligned} \tag{7}$$

E-field correction

Following Miller (DocDB 11082) and Nguyen (DocDB 12047), the relative contribution of the E-field to ω_a is

$$C_e = \left(1 - \frac{1}{a_\mu}\frac{m^2}{p^2}\right)\frac{\beta E_r}{cB} \sim \left(1 - \frac{m^2}{a_\mu p_0^2}(1 - 2\frac{\Delta p}{p})\right)\frac{\beta E_r}{cB} \tag{8}$$

If $\frac{m^2}{p_0^2} = a_\mu$, that is if we choose the magnetic field so that the magic momentum muon is at the magic radius, then

$$C_e \approx 2\frac{\Delta p}{p}\frac{\beta E_r}{cB}$$

Substitution of Equation 7 gives the correction to ω_a due to fractional momentum offset δ and betatron amplitude $x_{\beta0}$ as

$$\begin{aligned}
C_e(\delta, x_{\beta0}) &\approx 2\delta\frac{\beta k}{cB}\left(\eta\delta + \frac{1}{2\rho_0}((\eta\delta)^2 + \frac{1}{2}x_{\beta0}^2)\right) \\
&\approx 2\frac{\beta k}{cB}\left(\eta\delta^2 + \frac{1}{2\rho_0}(\eta^2\delta^3 + \frac{1}{2}x_{\beta0}^2\delta)\right)
\end{aligned}$$

Next we need to average $C_e(\delta, x_{\beta0})$ over the entire momentum and CBO distribution.

$$C_e(\delta, x_{\beta0}) \approx 2\frac{\beta k}{cB}\left(\frac{x_e^2}{\eta} + \frac{1}{2\rho_0}(\frac{x_e^3}{\eta} + \frac{1}{2}x_{\beta0}^2\frac{x_e}{\eta})\right)$$

where $\langle\delta\rangle = \langle x_e\rangle/\eta$. If $\langle\delta\rangle = \langle x_e\rangle/\eta = 0$, then

$$\langle C_e(\delta, x_{\beta0})\rangle = 2\frac{\beta k}{cB}\eta\langle\delta^2\rangle$$

and there is no contribution from the sextupole or path length terms. If $\langle\delta\rangle \neq 0$ then the relative adjustment to the E-field correction is

$$\text{relative} \sim \frac{\langle x_e\rangle}{\langle x_e^2\rangle}\frac{\langle x_{\beta 0}^2\rangle}{2\rho_0} = \frac{\eta\langle\delta\rangle}{\eta^2\langle\delta^2\rangle}\frac{\langle x_{\beta 0}^2\rangle}{2\rho_0} \sim \frac{\langle\delta\rangle\langle x_{\beta 0}^2\rangle}{2\eta\rho_0\langle\delta^2\rangle}.$$

If we suppose δ is large (~ 0.001), $\langle\delta^2\rangle$ small, ($\sim 10^{-7}$), and $\langle x_{\beta 0}^2\rangle/\eta^2$ is large ($\sim (0.04/(7)(8))^2 \sim 0.25\times 10^{-4}$), then the fractional error in the estimate is ~ 0.1, namely a 10% correction to the correction.

Now let's return to Equation 8 but this time we suppose that the distribution is not centered at the magic radius and choose p_0 according to the Miller/Nguyen rule to minimize the E-field correction.

Then

$$C_e \sim \left(1 - \frac{m^2}{a_\mu p_0^2}(1-2\frac{\Delta p}{p_0})\right)\frac{\beta E_r}{cB} \sim \left(1 - \frac{m^2}{a_\mu p_0^2}(1-2\frac{\Delta p}{p_0})\right)\frac{\beta k}{cB}\left(\eta\delta + \frac{1}{2\rho_0}((\eta\delta)^2 + \frac{1}{2}x_{\beta 0}^2)\right)$$

With Miller's definition of $\alpha = \frac{m^2}{p_0^2} - a_\mu$, and his finding that $\alpha = 2a_\mu\frac{\langle p-p_0\rangle}{p_0} = 2a_\mu\langle\delta\rangle$ minimizes the E-field correction when the average momentum is not p_0, the correction becomes

$$C_e \sim 2\left[-\langle\delta\rangle + (1+2\langle\delta\rangle)\delta\right]\frac{\beta k}{cB}\left(\eta\delta + \frac{1}{2\rho_0}((\eta\delta)^2 + \frac{1}{2}x_{\beta 0}^2)\right)$$

$$C_e \sim 2\left[-\langle\delta\rangle\eta\delta + \eta(1+2\langle\delta\rangle)\delta^2 - \frac{\langle\delta\rangle - \delta}{2\rho_0}(\eta\delta)^2 + \frac{1}{4\rho_0}(\delta - \langle\delta\rangle + \langle\delta\rangle\delta x_{\beta 0}^2\right]\frac{\beta k}{cB}$$

$$\langle C_e\rangle \sim 2\left[-\eta(\langle\delta\rangle^2 - \langle\delta^2\rangle) + \frac{1}{4\rho_0}(\langle\delta\rangle^2\langle x_{\beta 0}^2\rangle\right]\frac{\beta k}{cB}$$

The relative contribution to the E-field correction due to the finite betatron amplitude is

$$\Delta C_e/C_e \sim \frac{\langle x_{\beta 0}^2\rangle}{4\rho_0\eta}\left(\frac{\langle\delta\rangle^2}{\langle\delta\rangle^2 - \langle\delta^2\rangle}\right)$$

Is this important? Probably not. (Remember that $\eta \sim \rho_0/(1-n)$). Note however that the hypothesized sextupole component discussed above has opposite sign to and partially cancels the contribution from the path length effect. The real sextupole component may be somewhat different. Also a magnetic sextupole will similarly enhance (or at least complicate) the dependence of the E-field correction on betatron amplitude.

Multipoles Fitted to 3-D Field Map

The electric field along the horizontal axis ($y = 0$) is given by

$$E_x - iE_y = (b_n - ia_n)\frac{x^n}{r_0^n} \tag{9}$$

where $r_0 = 0.045$ m and a_n, b_n are given in Table 1. The multipoles are computed as a fit to an azimuthal slice of the 3D Opera field map of the quads[3]. The fit is for a 'horizontally' pure basis[1][2] of Mcmillan functions. Figure 3 shows the horizontal electric field in the midplane. The values from the Opera map, and from the multipole expansion are superimposed and evidently are in excellent agreement.

We can compare the fitted sextupole-like coefficient to our 'guess' discussed above. We found a solution to the Laplacian in the curved system as Equation 3. The ratio of the coefficients of the quadratic

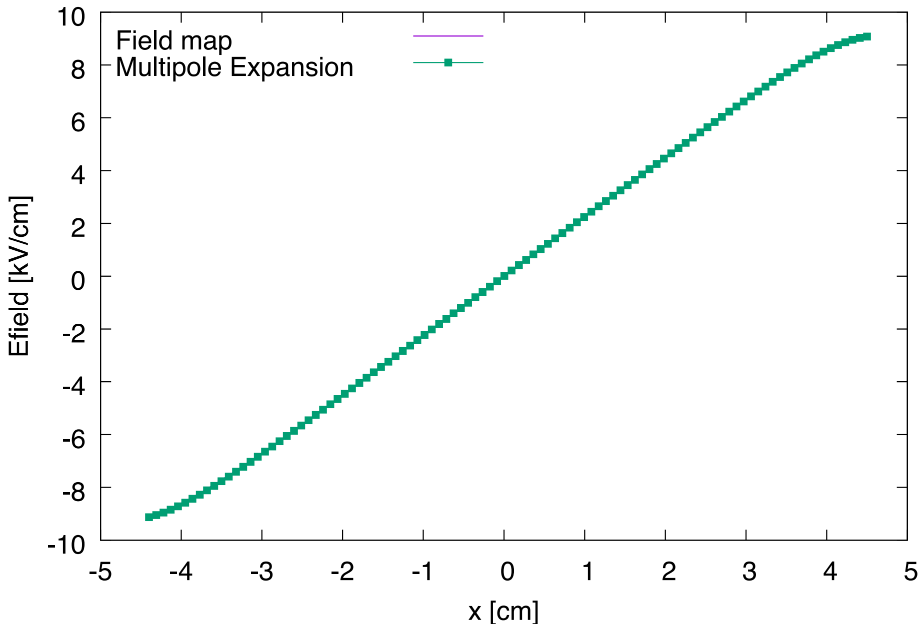

Figure 2: Electric field along the x-axis in the midplane (y=0). The green points

Figure 3: are computed from the multipole expansion using Equation 9 and the coefficients in Table 1. The 'purple' curve (hidden by the green points) are the values from the field map.

Table 1:

Multipole	Normal (b_n)	Skew (a_n)
1	1.01609E+06	-1.19899E+00
2	-2.71281E+03	7.71402E+00
3	-1.45524E+04	-1.50954E+01
4	-6.90244E+02	1.46521E+00
5	-5.23865E+03	4.35213E+01
6	1.00671E+02	-4.83666E+01
7	1.21107E+03	7.75800E+01
8	-1.43120E+02	4.99963E+00
9	-9.02621E+04	5.20048E+00
10	2.87638E+02	-3.47144E+01
11	5.36519E+03	3.43053E+01
12	1.05747E+02	2.39598E+00
13	8.00742E+02	-9.14390E+00

and linear terms is $r_{hyp} - \frac{1}{2\rho_0} = -0.0703\text{m}^{-1}$. But that 'hypothetical' solution is not unique. After all, while we insisted on a form that satisfies Maxwell's equations, we made no attempt to also satisfy the boundary conditions. The ratio based on the fit to the Opera field map, that satisfies both Maxwell and the boundary conditions, taking the values from the table is $r_{fit} = \frac{b_2}{b_1 r_0} = -2.71281\times 10^3/1.01609\times 10^6 = -0.0593\text{m}^{-1}$, within 16% of our guess.

As was shown above,the effect of the sextupole-like component of the quadrupole, and the asymmetry of the path length about the magic radius, is that the E-field correction depends on betatron amplitude. From Equations 4- 6 we see that the sextupole-like component and the path length contribute with opposite sign, and the amplitude of the sextupole-like component is about 1/2 of the path length piece.

Bibliography

[1] D. Sagan, “Bmad Manual” https://www.classe.cornell.edu/bmad/manual.html, section 14.3 pp. 254-256.

[2] E. M. McMillan,“Multipoles in Cylindrical Coordinates,âĂİ Nucl. Instrum. Meth. 127, 471 (1975)

[3] W. Wu, “The 3D electric field map from OPERA-3D”, E989 Note 117: docdb 8162-v1, 6 Sept. 2017