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June 26, 2018

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September 8, 2020

Closed Orbit Distortion

Contribution from Electric Field

We have

$$\begin{aligned}\Delta\omega_E &= -\frac{e}{mc}a_\mu(-2\frac{\Delta p}{p}\beta_\gamma E_\rho) \\ \rightarrow \frac{\Delta\omega_E}{\omega_a} = C_e &= \frac{-2\langle\frac{\Delta p}{p}\beta_\gamma E_\rho\rangle}{B}\end{aligned}$$

We can write $\frac{\Delta p}{p} = x_e\eta$, where x_e is the momentum dependent displacement of the closed orbit from the magic radius. The radial electric field is given by

$$E_\rho = n\left(\frac{\beta_\gamma B}{R_0}\right)x_e$$

so that

$$\langle C_e\rangle = -2\beta_\gamma^2 n(1-n)\frac{\langle x_e^2\rangle}{R_0^2}$$

where we have used $\eta = R_0/(1-n)$.

What can go wrong? We measure the revolution frequency.

$$\begin{aligned}\Delta T &= 2\pi\langle x(\phi)\rangle/v \\ x(\phi) &= \eta(\phi)\frac{\Delta p}{p} \\ \Delta T v &= 2\pi\langle\eta\rangle\frac{\Delta p}{p} \\ 2\pi\frac{\Delta\omega}{\omega^2}v &= 2\pi\langle\eta\rangle\frac{\Delta p}{p} \\ \frac{v\Delta\omega}{\omega^2} &= \frac{\Delta p}{p}\langle\eta(\phi)\rangle \\ R\frac{\Delta\omega}{\omega} &= \frac{\Delta p}{p}\langle\eta(\phi)\rangle \\ \frac{\Delta p}{p} &= \frac{R}{\langle\eta\rangle}\frac{\Delta\omega}{\omega}\end{aligned}$$

We see that to convert frequency offset into momentum offset, we need the dispersion averaged around the ring. Now our usual approximation of the electric field is

$$E_\rho = n\left(\frac{\beta_\gamma B}{R}\right)x_e = kx_e$$

and including the possibility of a closed orbit error

$$x_e = x_c(\phi) + \eta(\phi)\frac{\Delta p}{p}$$

Then we can write

$$\frac{B}{\beta_\gamma}C_e = -2\left(\frac{\Delta p}{p}\right)\left(\langle x_c(\phi)k(\phi)\rangle + \frac{\Delta p}{p}\langle \eta(\phi)k(\phi)\rangle\right)$$

So we need

$$\langle \eta \rangle = \frac{1}{2\pi}\oint \eta(\phi)d\phi$$
$$\langle \eta k \rangle = \frac{1}{2\pi}\oint \eta(\phi)k(\phi)d\phi$$
$$\langle x_c k \rangle = \frac{1}{2\pi}\oint x_c(\phi)k(\phi)d\phi$$

Magnetic field is periodic.

$$B = \sum_r B_r \cos(r\theta)$$

$r = 0$ establishes closed orbit. $r > 0$ leads to closed orbit distortion. Evaluate closed orbit and dispersion

$$\begin{aligned}
\eta(s) &= \frac{\sqrt{(\beta(s)}}{2\sin\pi\nu}\oint \frac{e}{p}\sum_r B_r \cos(r\theta(s'))\sqrt{\beta(s')}\cos(\phi(s)-\phi(s')-\pi\nu)ds' \\
d\phi &= ds/\beta, \quad \phi = \nu\theta \\
\eta(\theta) &= \frac{\sqrt{\beta(\theta)}}{2\sin\pi\nu}\int_{\theta-2\pi}^{\theta}\frac{e}{p}\sum B_r\cos(r\theta')\nu\sqrt{\beta^3(\theta')}\cos(\nu\theta-\nu\theta'-\pi\nu)d\theta'
\end{aligned}$$

If $n = 0$ and $\frac{e}{p}B_0 = 1/R_0$ and $\beta(\theta) = \beta$

$$\begin{aligned}
\eta(\theta) &= \frac{\nu}{2\sin\pi\nu}\frac{\beta^2}{R_0}\frac{\sin(\nu\theta-\nu\theta'-\pi\nu)}{-\nu}|_{\theta-2\pi}^{\theta} \\
\eta(\theta) &= \frac{\nu}{2\sin\pi\nu}\frac{\beta^2}{R_0}\frac{\sin(-\pi\nu)-\sin(\pi\nu)}{-\nu} \\
\eta(\theta) &= \frac{\beta^2}{R_0} = \frac{R_0}{1-n}
\end{aligned}$$

where $\beta = R_0/\sqrt{1-n}$. More generally the r^{th} moment of the dispersion

$$\begin{aligned}
\eta_r(\theta) &= \frac{\sqrt{\beta(\theta)}}{2\sin\pi\nu}\int_{\theta-2\pi}^{\theta} b_r\cos(r\theta')\nu\beta^{3/2}(\theta')\left(\cos(\nu\theta-\nu\theta'-\pi\nu)\right)d\theta' \\
\eta_r(\theta) &= \frac{\sqrt{\beta(\theta)}}{2\sin\pi\nu}\int_{\theta-2\pi}^{\theta} b_r\nu\beta^{3/2}(\theta')\frac{1}{2}\left(\cos(r\theta'+\nu\theta-\nu\theta'-\pi\nu)+\cos(r\theta'-\nu\theta+\nu\theta'+\pi\nu)\right)d\theta' \\
\eta_r(\theta) &\approx \frac{\beta^2}{2\sin\pi\nu}b_r\nu\frac{1}{2}\left(\frac{1}{r-\nu}\sin(r\theta'+\nu\theta-\nu\theta'-\pi\nu)+\frac{1}{r+\nu}\sin(r\theta'-\nu\theta+\nu\theta'+\pi\nu)\right)|_{\theta-2\pi}^{\theta} \\
\eta_r(\theta) &\approx \frac{\beta^2}{2\sin\pi\nu}b_r\nu\frac{1}{2}\left(\frac{1}{r-\nu}\left(\sin(r\theta-\pi\nu)-\sin(r\theta+\pi\nu)\right)+\frac{1}{r+\nu}\left(\sin(r\theta+\pi\nu)-\sin(r\theta-\pi\nu)\right)\right) \\
\eta_r(\theta) &\approx \frac{\beta^2}{2\sin\pi\nu}b_r\nu\left(\frac{-\cos(r\theta)\sin(\pi\nu)}{r-\nu}+\frac{\cos(r\theta)\sin(\pi\nu)}{r+\nu}\right) \\
\eta_r(\theta) &\approx \frac{1}{2}\beta^2 b_r\cos(r\theta)\left(-\frac{\nu}{r-\nu}+\frac{\nu}{r+\nu}\right) \\
\eta_r(\theta) &\approx \beta^2 b_r\cos(r\theta)\left(\frac{-\nu^2}{r^2-\nu^2}\right) \qquad (1)
\end{aligned}$$

where $b_r = \frac{e}{p} B_r$ with dimensions of inverse length and in the third line we assume β is constant. Equation 1 also gives the closed orbit distortion for the on energy particle for the r^{th} multipole ($r > 0$) of the magnetic field. For $r = 0$ 1 becomes

$$\eta = \frac{\beta^2}{R_0} = \frac{R_0}{1-n}$$

where $b_0 = 1/R_0$ and $\beta = R_0/\sqrt{1-n}$. For $r = 1$, looks pretty much like the calculation with the measured field so most of field error is $n = 1$ and $b_r \sim 5.6 \times 10^{-6}\mathrm{m}^{-1}$ or about 40 ppm.

The closed orbit distortion for the on momentum particle is $x_r(\theta) = \eta(\theta)$.

The next step is to compute $\langle \eta k \rangle$. We can write the azimuthal dependence of the electric field, just like the magnetic. Due to the four fold symmetry

$$k(\theta) = \sum k_m \cos(4m\theta + \phi_m)$$

where $k_1 = k_0$ so that the first 2 terms give

$$k(\theta) = k_0(1 + \cos(4\theta + \phi_m))$$

Then

$$\begin{aligned} \langle \eta k \rangle_{r,m} &= 2\beta^2 b_r \left(\frac{-\nu^2}{r^2 - \nu^2} \right) \int \sum_{m=0} k_m \cos(4m\theta + \phi_m) \cos(r\theta) d\theta \\ \langle \eta k \rangle_{r,m} &= \beta^2 b_r \left(\frac{-\nu^2}{r^2 - \nu^2} \right) \sum_m k_m \delta_{4m,r} \end{aligned}$$

We find that only $r = 0, 4, 8, \ldots$ contribute to the average. We can try to compare the contribution from the $m = 1, r = 4$ to the $m = r = 0$ part.

$$\frac{\langle \eta k \rangle_{4,1}}{\langle \eta k \rangle_{0,0}} = \frac{-\nu^2}{16 - \nu^2} \frac{b_4}{b_0} \frac{k_1}{k_0}$$

Keeping the leading terms

$$k(\theta) = k_0 + k_1 \cos(4\theta + \phi_m)$$

it appears that $k_0 = k_1$ so that k varies between 0 and $2k_0$.

$$\frac{\langle \eta k \rangle_{4,1}}{\langle \eta k \rangle_{0,0}} = \frac{\nu^2}{16 - \nu^2} \frac{b_4}{b_0}$$

Fitting to the measured azimuthal field on the closed orbit from the measured field we find that $b_4/b_0 \sim 4$ ppm and therefore

$$\frac{\langle \eta k \rangle_{4,1}}{\langle \eta k \rangle_{0,0}} \sim 233 \text{ ppb}$$

That is, the correction due to the distortion of the dispersion function is less than 1 part in a million.

Closed Orbit Distortion

As $x_r(\theta) = \eta(\theta)_r$ we know that

$$\langle x_r k\rangle = 0,\ r \neq 4$$

$$\langle x_4 k_1\rangle = \beta^2 b_4 \left(\frac{-\nu^2}{r^2-\nu^2}\right) k_1 \sim 10^{-5}\mathrm{m}^{-1} k_1$$

To what do we compare this contribution? Let's try

$$\frac{\langle x_c k\rangle}{(\Delta p/p)\langle \eta k\rangle} = \frac{10^{-5}}{(\Delta p/p)\eta_0} \sim \frac{10^{-5}}{8} = \frac{1.3\times 10^{-6}}{\Delta p/p}$$

If $\Delta p/p < 1\times 10^{-6}$ the closed orbit distortion dominates. The correction for such a momentum offset is

$$\begin{aligned} C_e(\Delta p/p) &= -2\beta^2 n(1-n)\frac{\langle x_e^2\rangle}{R_0^2} \\ &= -2\beta^2 n(1-n)\frac{\langle(\eta\Delta p/p)^2\rangle}{R_0^2} \\ &= -2\beta^2 n\frac{\langle(\Delta p/p)^2\rangle}{1-n} \\ &\sim -2\frac{n}{1-n}(\Delta p/p)^2 \\ &\sim -2\frac{0.11}{1-0.11}(10^{-6})^2 \\ &\sim -.25\times 10^{-12} \end{aligned}$$

or -0.25×10^{-3} ppb.

What about that earlier approximation that β is constant around the ring. A better approximation is

$$\beta(\theta) = \beta_0 + \beta_4\cos(4\theta+\phi_\beta)$$

The term

$$\beta^{3/2}(\theta) \sim \beta_0^{3/2}(1+\frac{3}{2}\frac{\beta_4}{\beta_0}\cos(4\theta+\phi_\beta))$$

Compute the contribution to the dispersion due to the variations in β.

$$\eta_r(\theta) = \beta_0^2\frac{1+\frac{1}{2}\frac{\beta_4}{\beta_0}\cos(4\theta+\phi_\beta)}{2\sin\pi\nu}\int_{\theta-2\pi}^{\theta} b_r\frac{3\beta_4}{2\beta_0}\cos(r\theta')\nu\cos(4\theta'+\phi_\beta)\left(\cos(\nu\theta-\nu\theta'-\pi\nu\right)d\theta' \quad (2)$$

It looks like we will end up with terms proportional to $\cos(r\pm 4)$ and $\cos(r\pm 4\pm 4)$, neither of which will contribute to the integral of dispersion and electric field.

Bibliography

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