Simulation of E-field systematics and dependence on betatron amplitude

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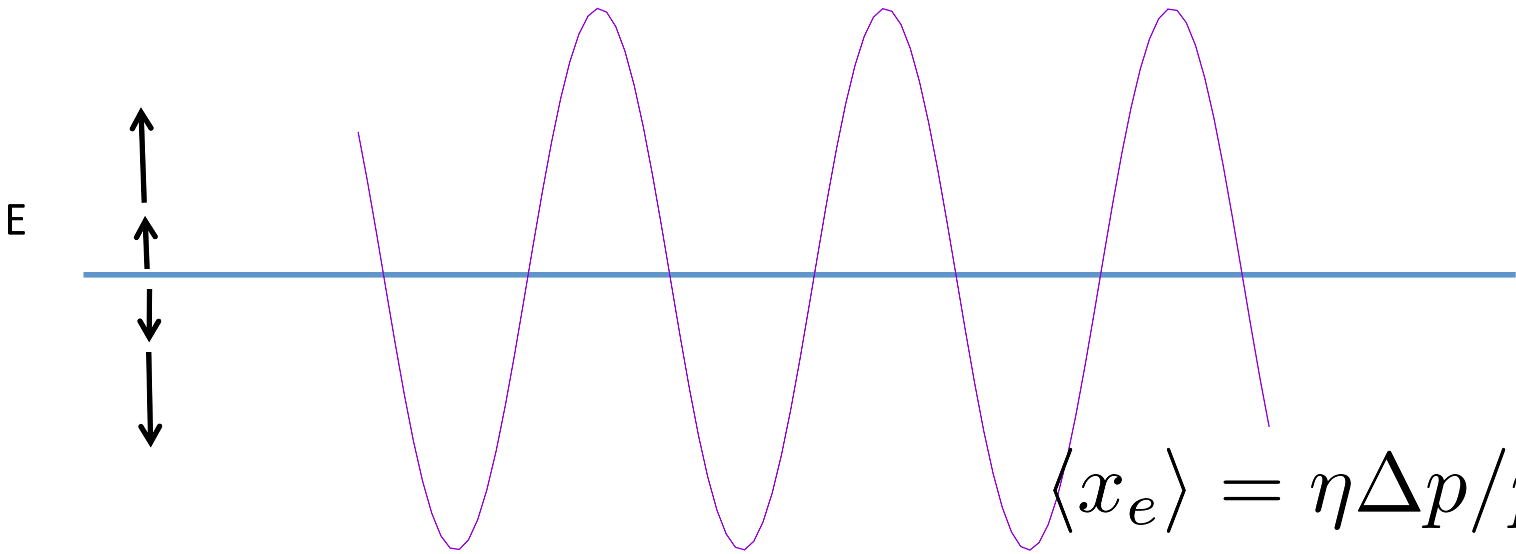
$$\vec{\omega}_a = -\frac{q}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{m^2}{p^2}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right]$$

E-field contribution

In an ideal cartesian geometry and quad field where the radial field is antisymmetric about the magic radius, the E-field correction is independent of betatron amplitude



$$\langle x_e \rangle = \eta \Delta p/p$$

In a curved geometry, the integrated E-field along the trajectory depends on betatron amplitude in two ways

1. Sextupole component of the quads
   - Sextupole component is symmetric about magic radius
   - Shifts the 'closed orbit'
2. Path length (asymmetric about magic radius)

$$\nabla V = \mathbf{E} \sim k \left((x - \frac{x^2}{\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}} \right)$$

$$\begin{aligned}
\langle E_r(s) \rangle &= k \langle \left(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2 \right) \rangle \\
&= \frac{k}{L} \int_0 \left(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2 \right) dl \\
&= \frac{k}{L} \int_0 \left(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2 \right) (1 + x_\beta/\rho_0) ds
\end{aligned}$$

sextupole

Path length

The average E-field for a muon with momentum $p_0 + \Delta p$ and betatron amplitude x_β is

$$\langle E_r \rangle \quad = \quad k\left(\eta\delta + \frac{1}{2\rho_0}((\eta\delta)^2 + \frac{1}{2}x_{\beta 0}^2) + \ldots\right)$$

$$(x_e = \eta\delta)$$

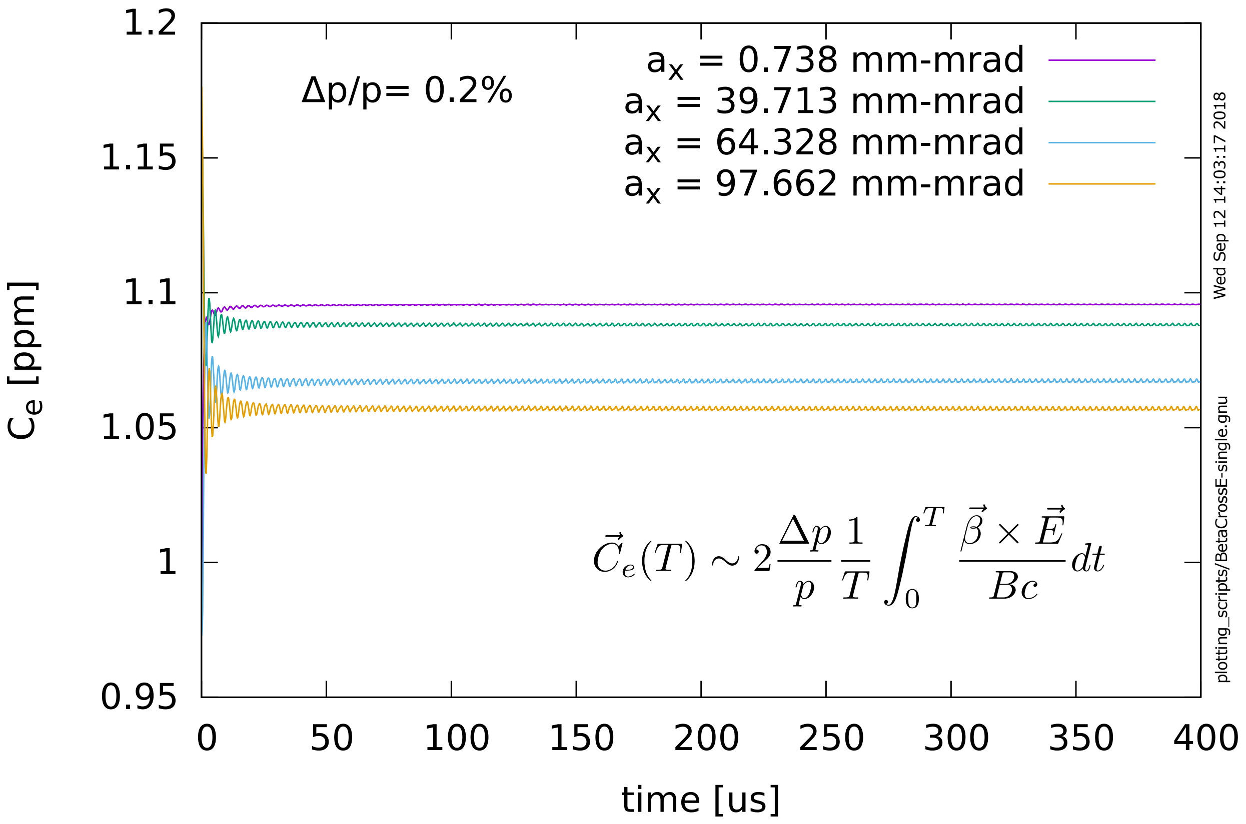
For positive momentum offset correction *increases* with betatron amplitude

E-field correction

$$\vec{C}_e(T) \sim 2\frac{\Delta p}{p}\frac{1}{T}\int_0^T \frac{\vec{\beta} \times \vec{E}}{Bc}dt$$

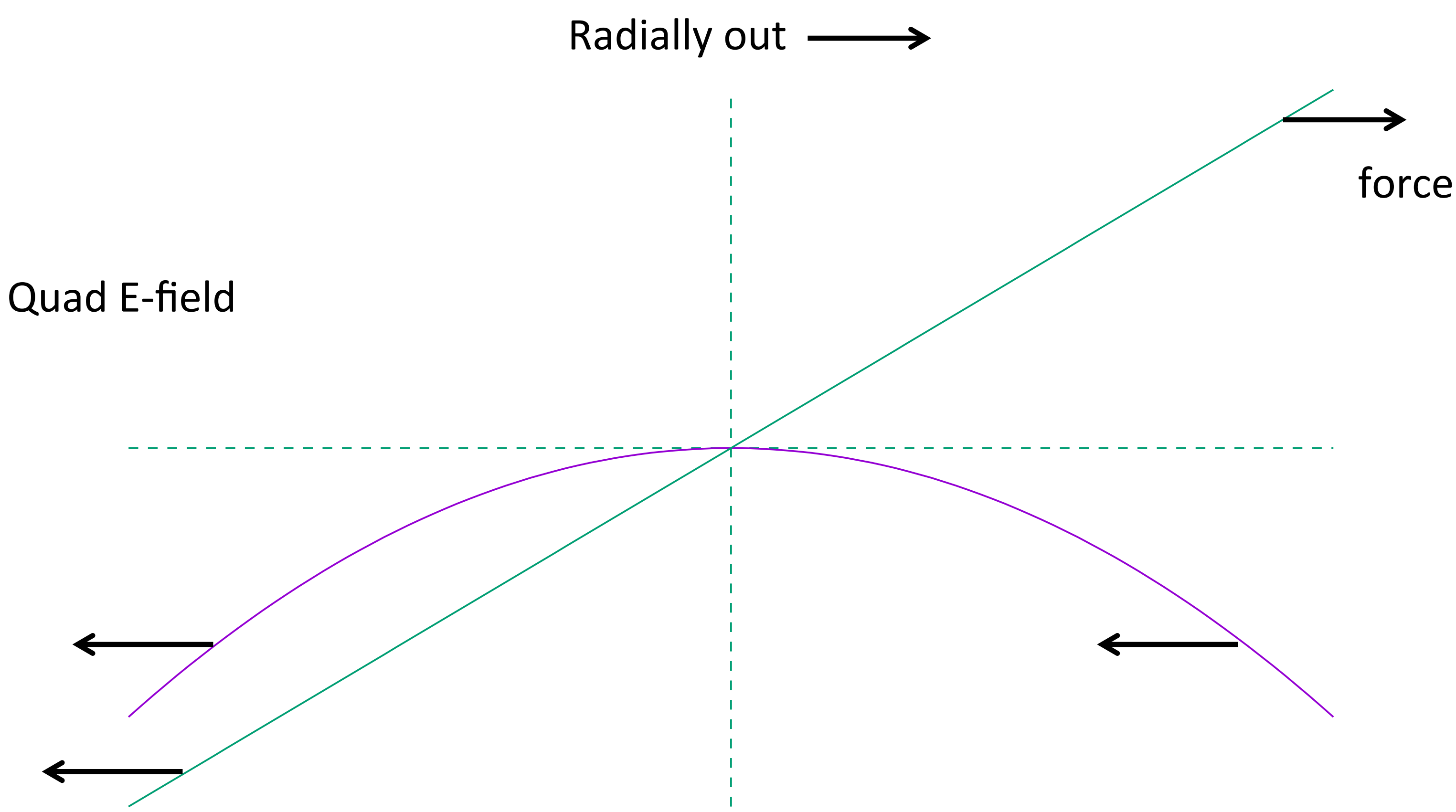
To compute correction in simulation integrate $\langle\vec{\beta} \times \vec{E}\rangle$ along the trajectory of the muon

=> E-field correction as a function of time, $C_e(t)$



- $C_e(t)$ oscillates with betatron frequency at early time
- $C_e(t)$ decreases with increasing betatron amplitude
- Contribution from betatron amplitude < 40 ppb

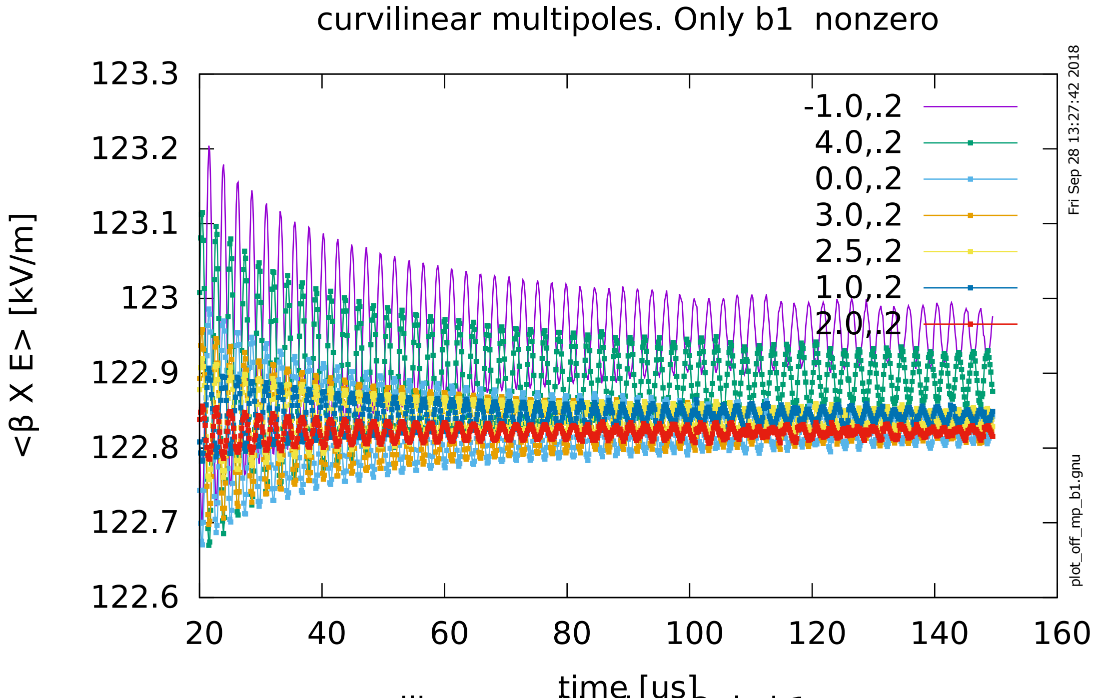
Correction *decreases* with amplitude



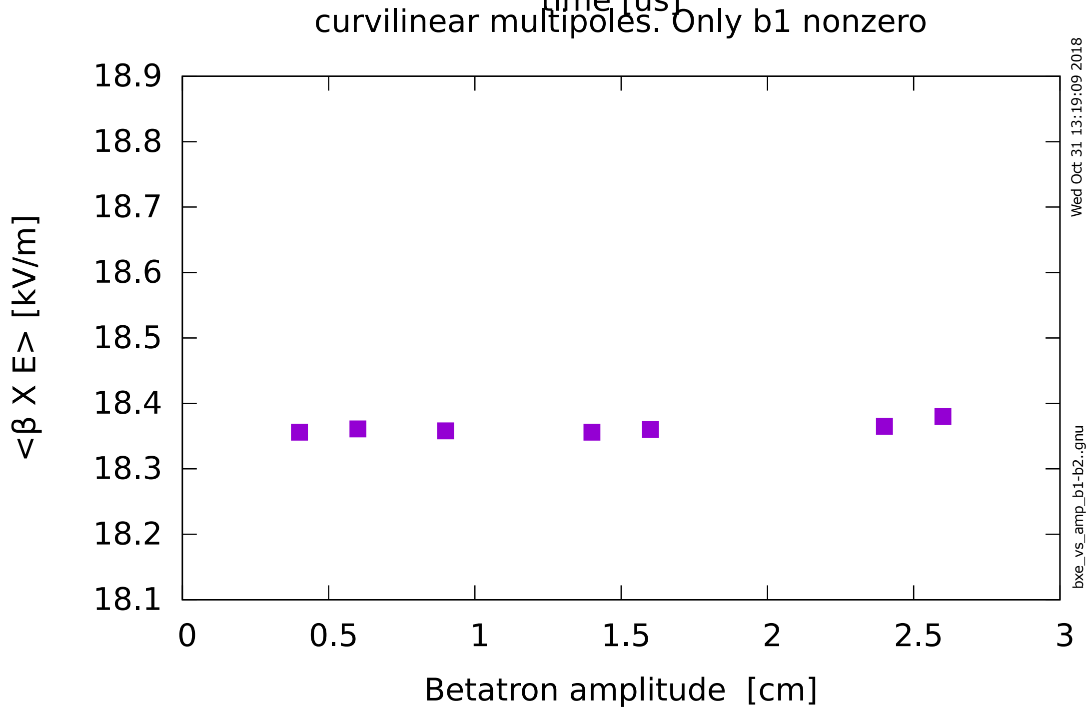
Sextupole component shifts <x> radially inward by

$$\langle F_x \rangle \sim k \frac{x_\beta^2}{2\rho}$$

Turn off sextupole component

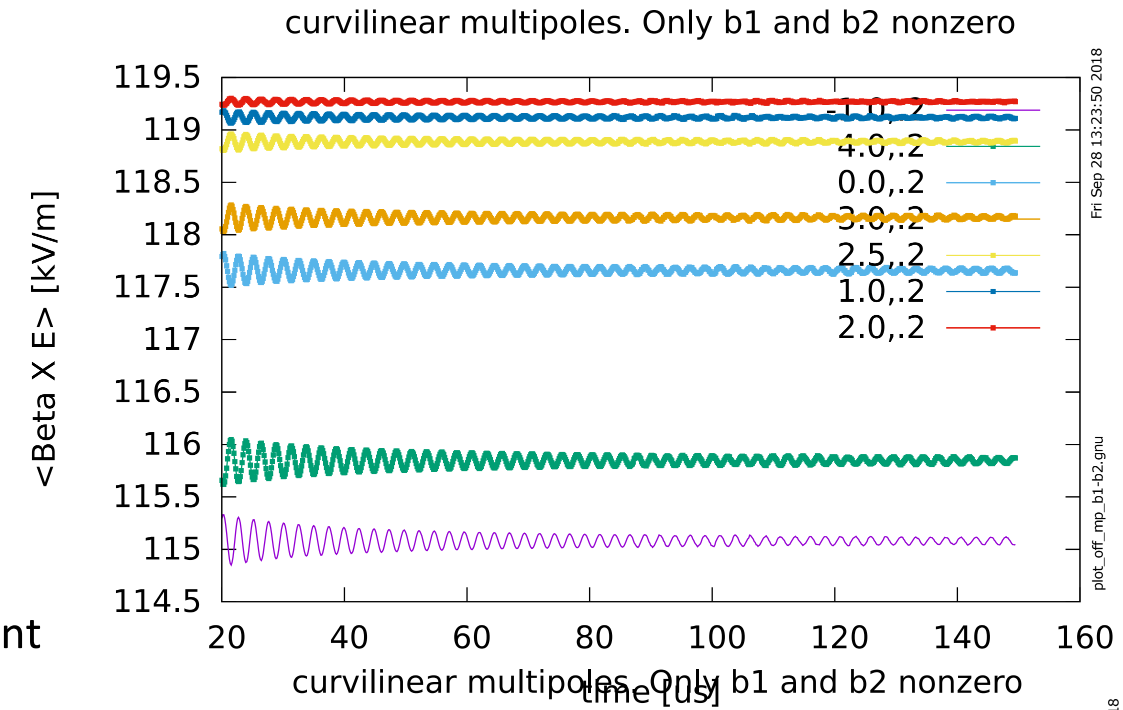


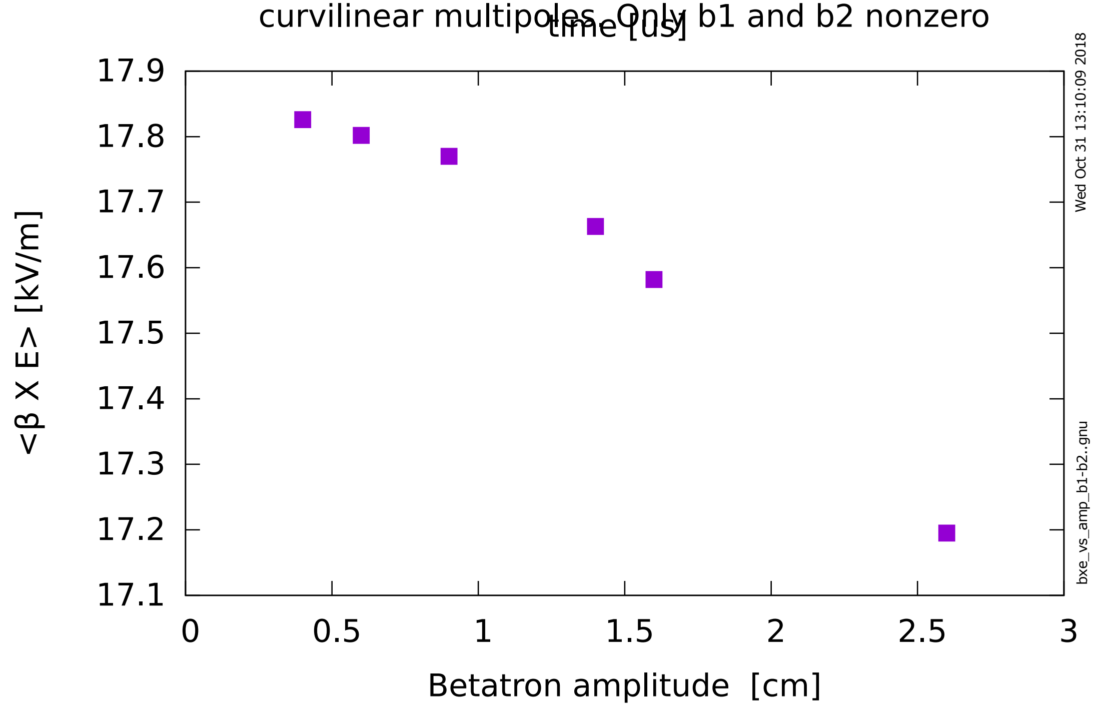
Correction increases with amplitude (path length effect)



Restore sextupole component

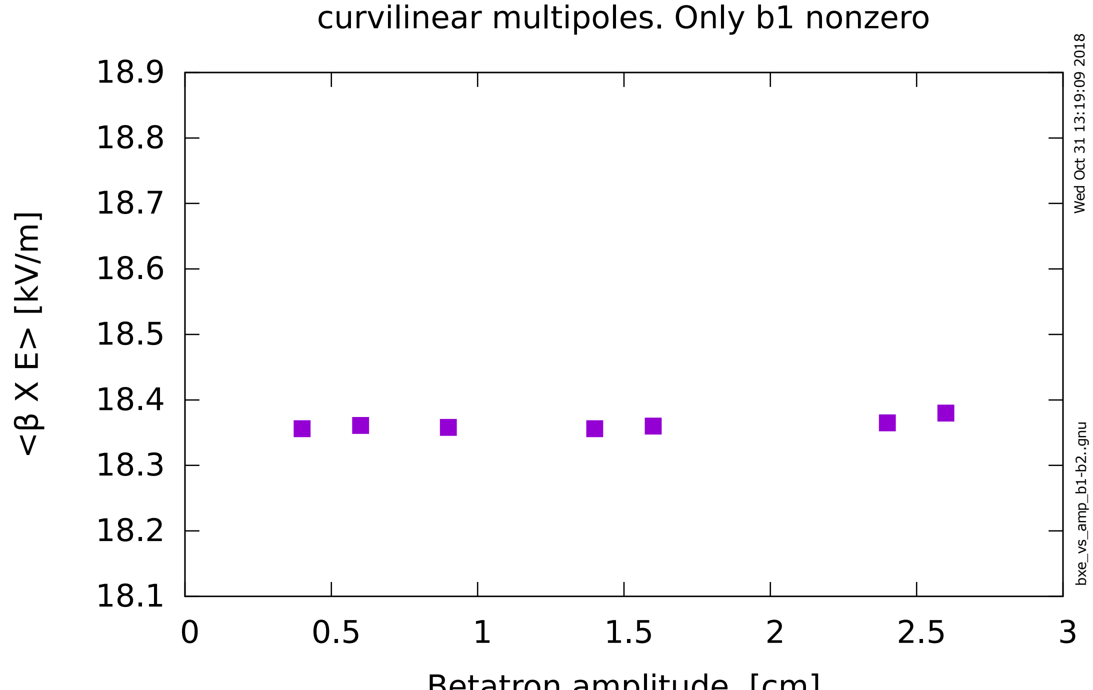
Correction *decreases* with amplitude (average orbit shifted radially inward)





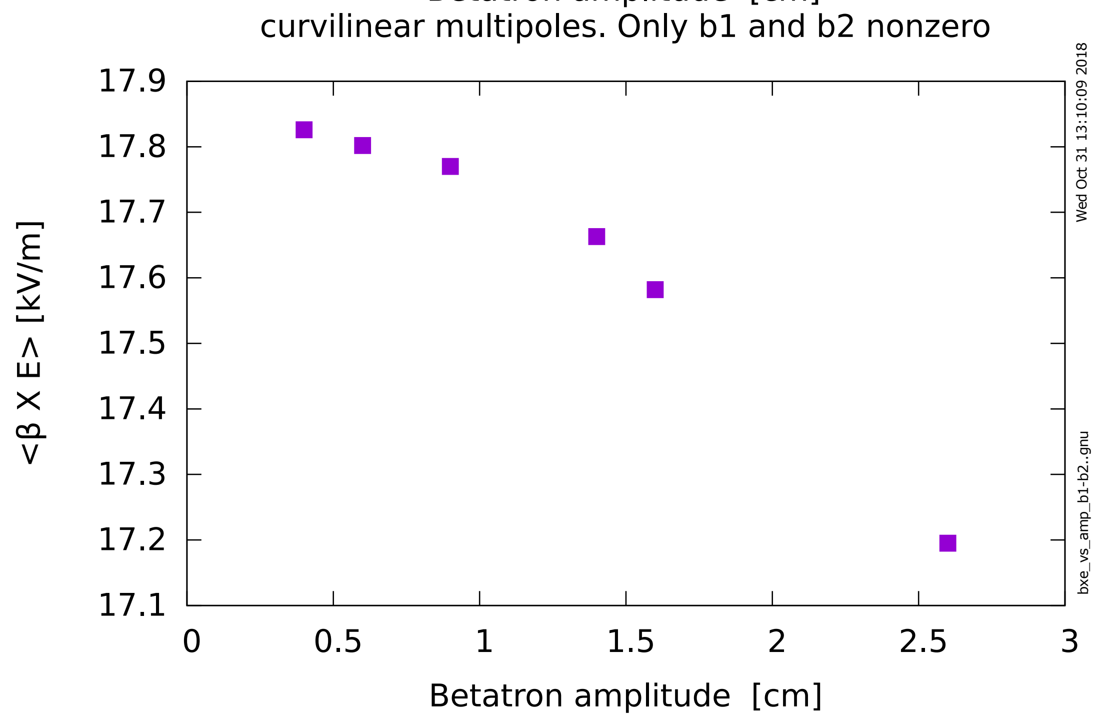
Dependence of E-field correction on Betatron amplitude with and without sextupole-like

$$\Delta p/p = 0.2\%$$



Efield correction decreases with betatron amplitude for $\Delta p/p > 0$

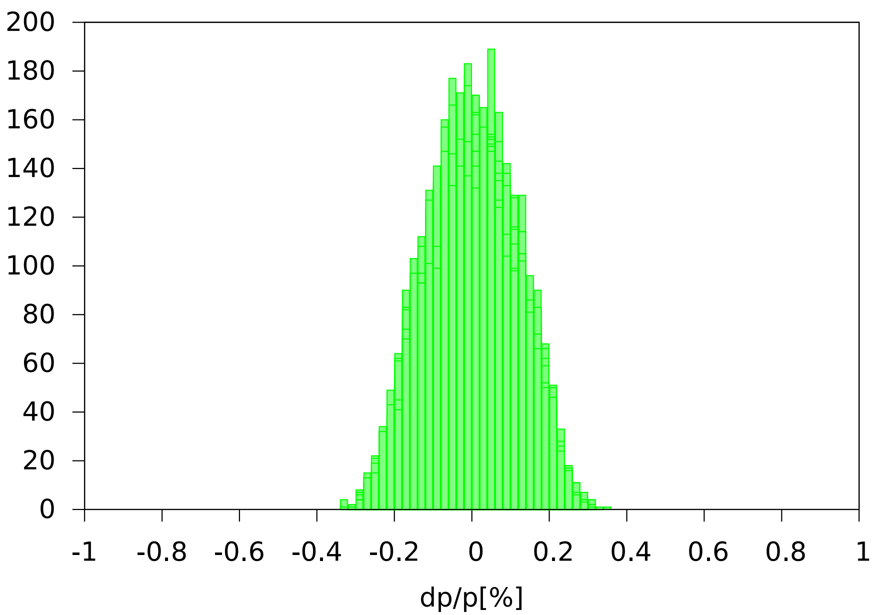
And increases for $\Delta p/p < 0$

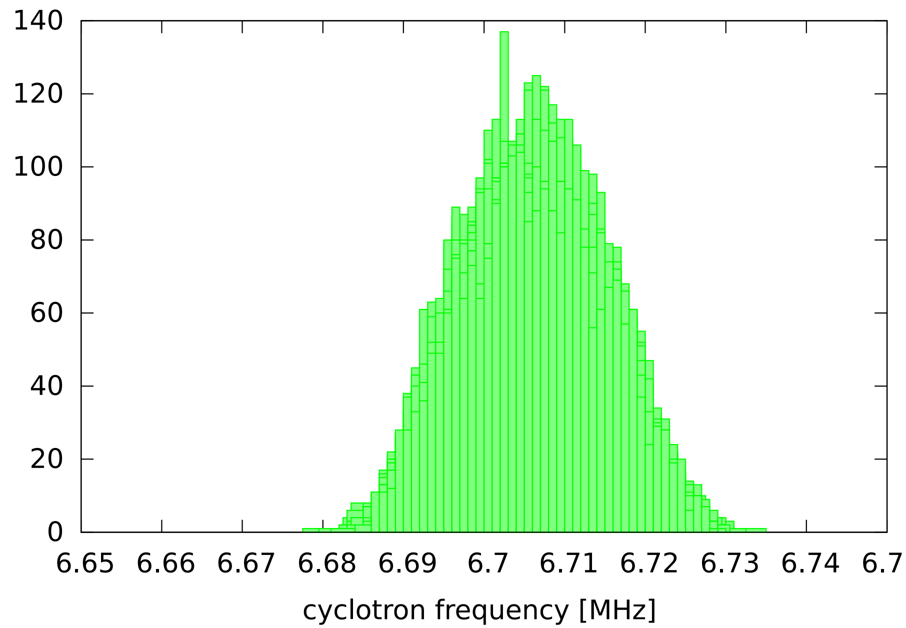


E-field correction estimated from Fast Rotation analysis and reconstruction of momentum distribution

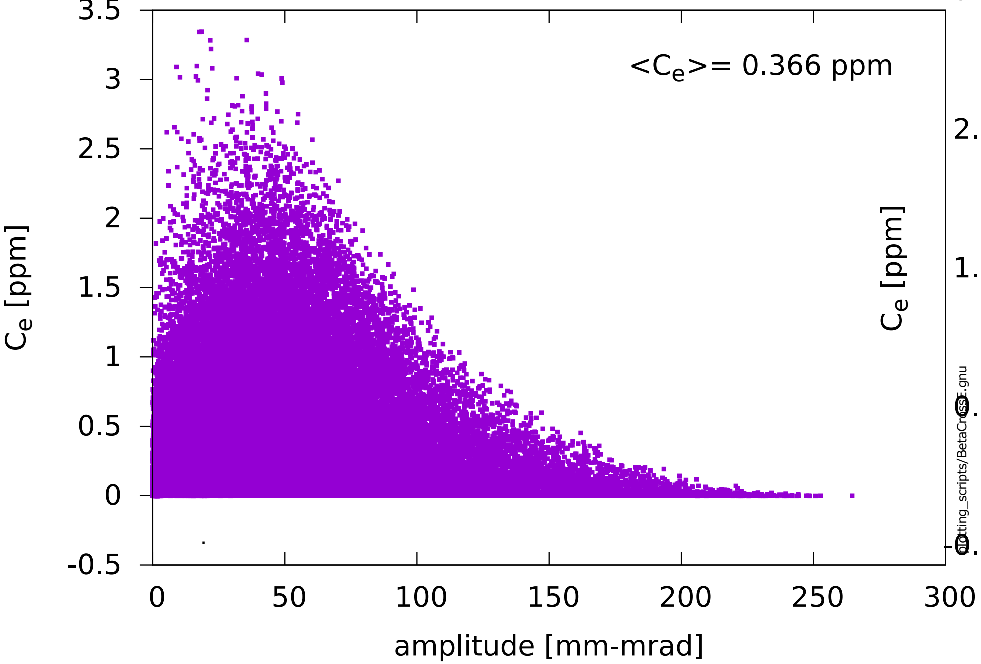
We plan to test with simulation

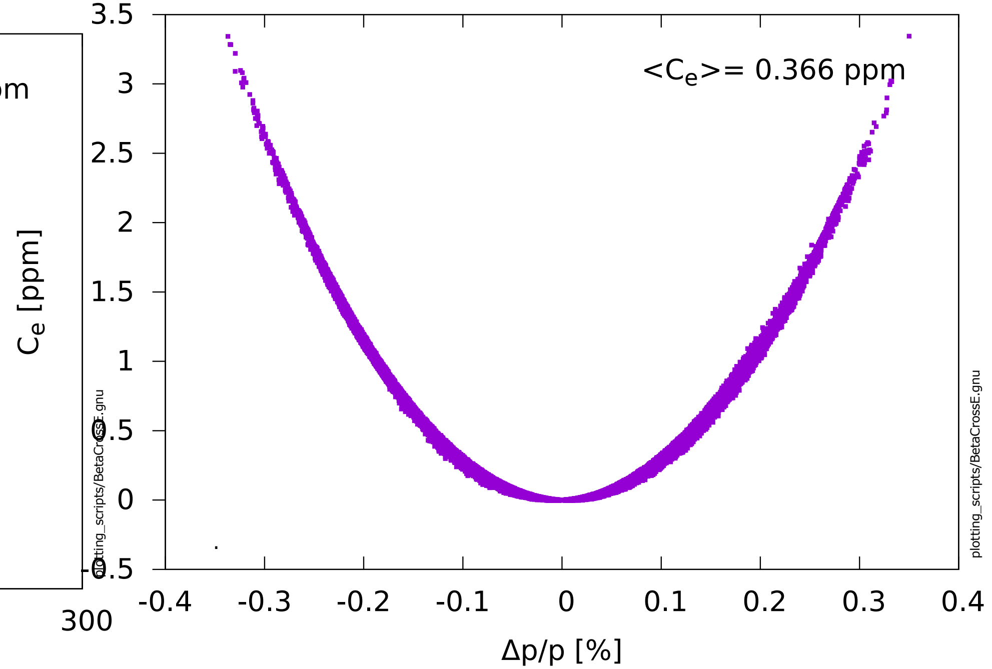
1. Generated and track a distribution and compute
    - Momentum and frequency distribution



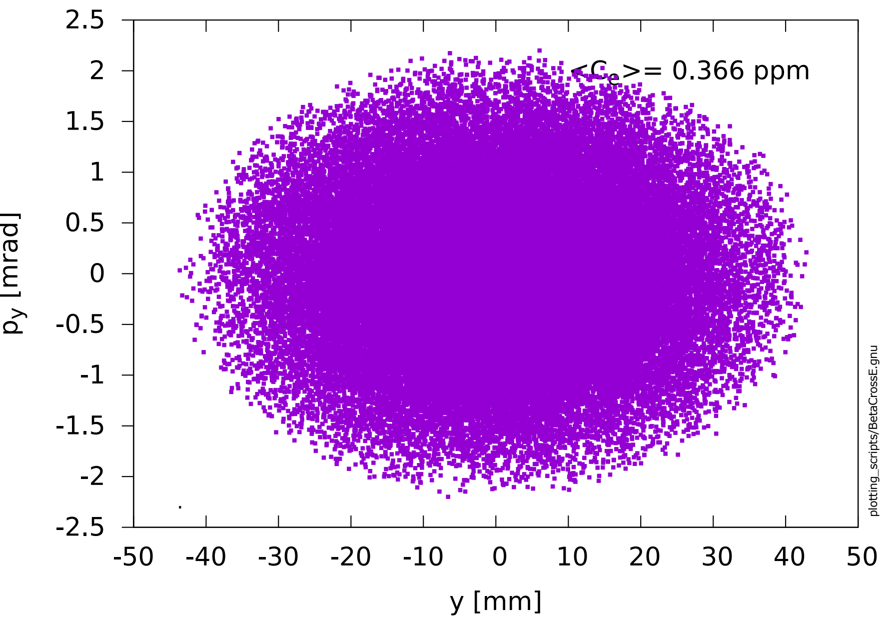


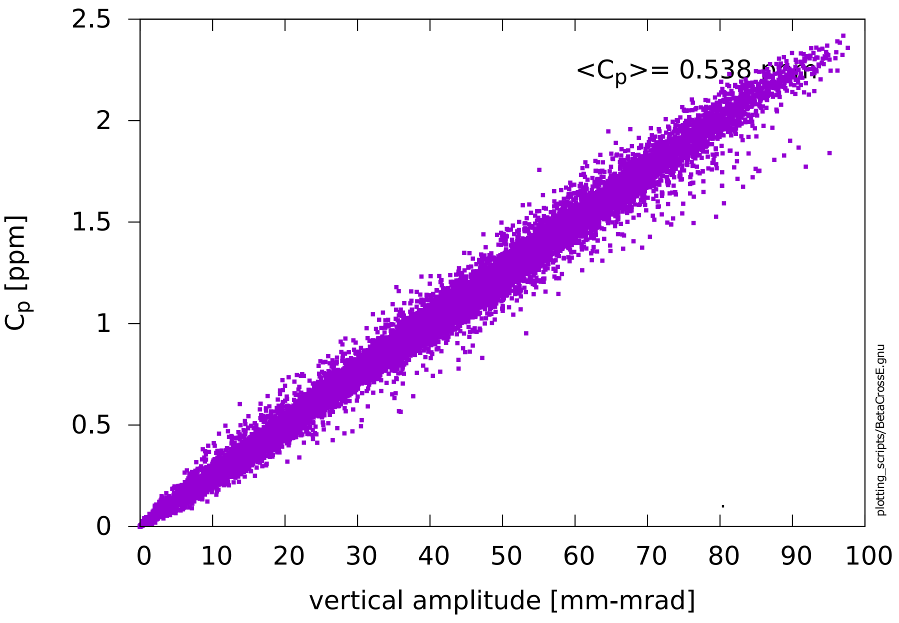
1. Generated and track a distribution and compute
    - Momentum and frequency distribution
    - E-field correction for each particle



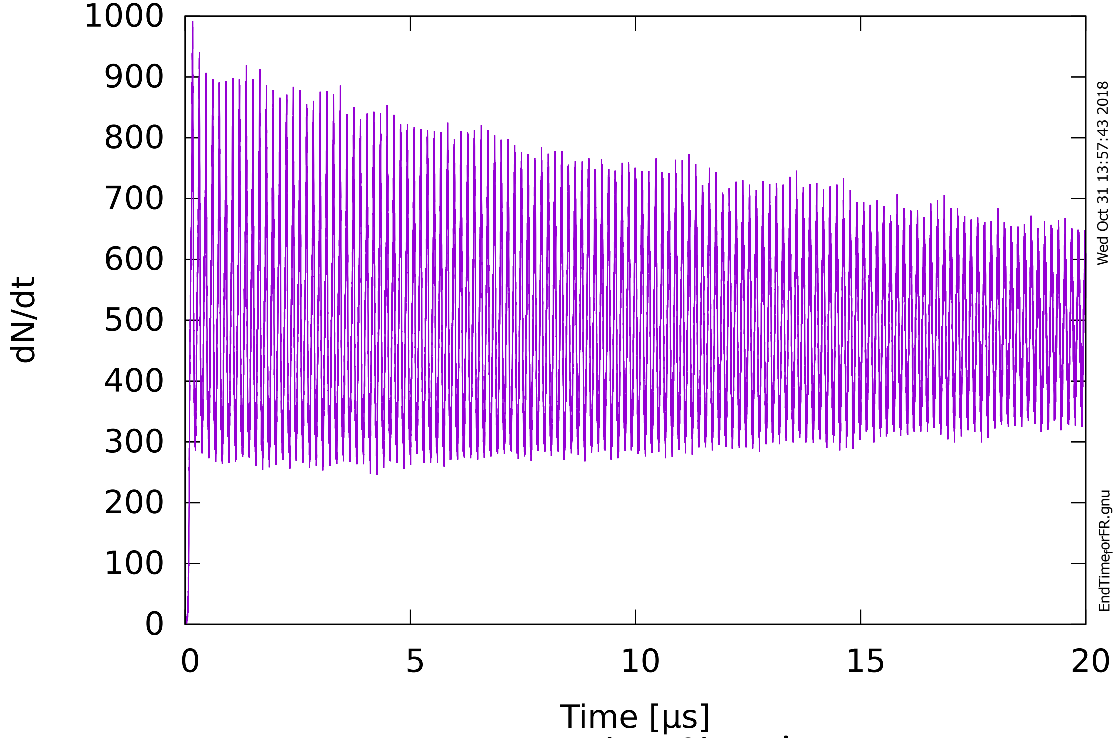


1. Generated and track a distribution and compute
    - Momentum and frequency distribution
    - E-field correction for each particle
    - Vertical phase space distribution and pitch correction





1. Generated and track a distribution and compute
   - Momentum and frequency distribution
   - E-field correction for each particle
   - Vertical phase space distribution and pitch correction
   - Fast rotation signal



Fast Rotation Signal

1. Generated and track a distribution and compute
    - Momentum and frequency distribution
    - E-field correction for each particle
    - Vertical phase space distribution and pitch correction
    - Fast rotation signal
2. Perform fast rotation analysis and see if we get the right answer.

Summary

- Effect of E-field and pitch is computed in tracking simulation by integrating (summing along the trajectory)

$$\vec{C}_e(T) \sim 2\frac{\Delta p}{p}\frac{1}{T}\int_0^T \frac{\vec{\beta}\times\vec{E}}{Bc}dt$$

$$\vec{C}_p(T) = \frac{1}{B}\frac{1}{T}\int_0^T (\vec{\beta}\cdot\vec{B})\vec{\beta}dt$$

- Explore dependence on betatron amplitude and quad nonlinearity
- Generate and track ‘realistic’ distributions − including E-field correction and fast rotation signal
    - Extract momentum distribution with FR-analysis
    - Compare E-field correction based on FR-analysis with the real thing

END

Backup

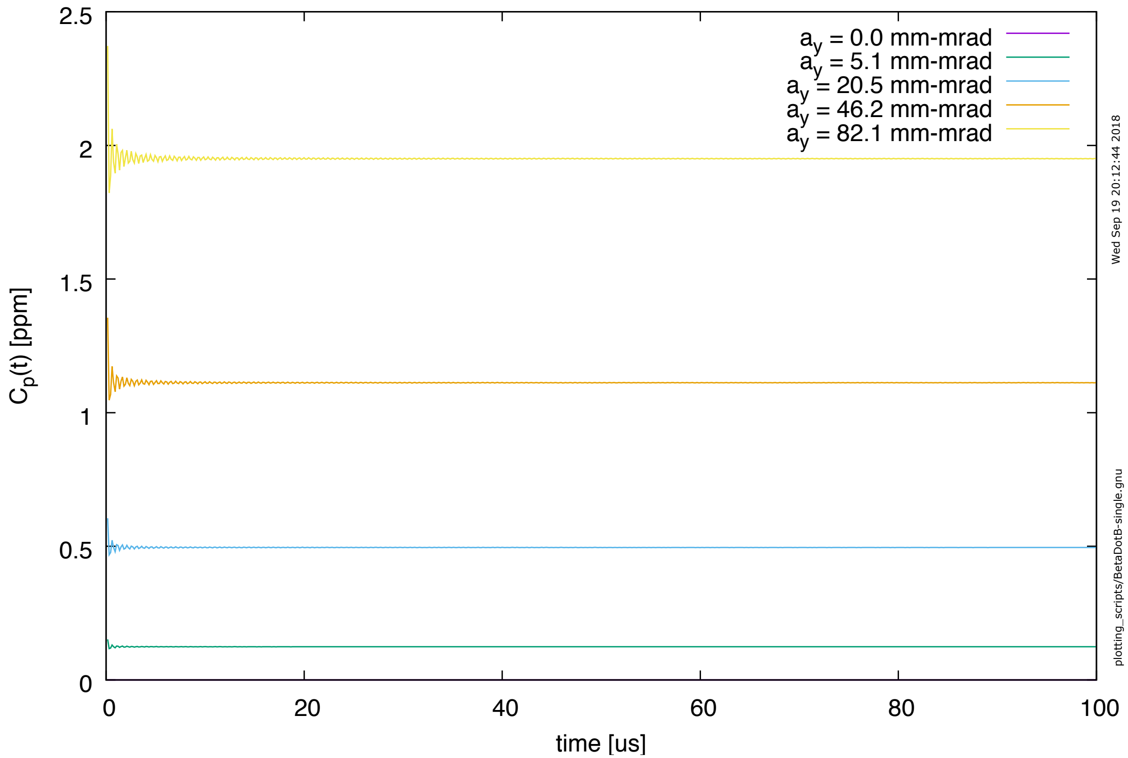
Pitch correction

$$\vec{C}_p(T) = \frac{1}{B}\frac{1}{T}\int_0^T (\vec{\beta}\cdot\vec{B})\vec{\beta}dt$$

How does pitch correction depend on betatron amplitude?

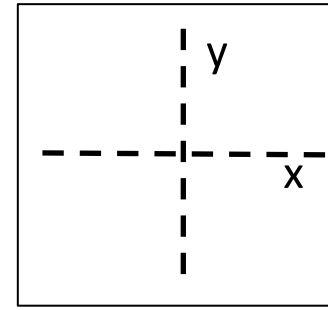
The initial displacements and amplitudes are

1. $y = 0$, $a_y = 0.0$ mm-mrad
2. $y = 1$ cm, $a_y = 5.12$ mm-mrad
3. $y = 2$ cm, $a_y = 64.3$ mm-mrad
4. $y = 3$ cm, $a_y = 46.2$ mm-mrad
5. $y = 4$ cm, $a_y = 82.1$ mm-mrad



Consider the Laplacian in 2 dimensions and cartesian coordinates

$$\nabla^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) V(x, y)$$



A solution that corresponds to the perfect quadrupole is

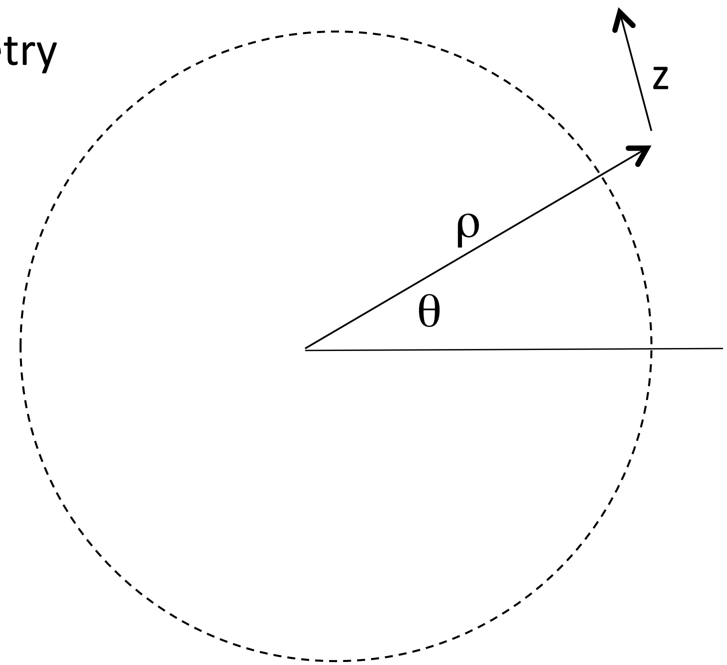
$$V(x, y) = \frac{1}{2} k (x^2 - y^2)$$

Then the divergence gives the E-field, linear in x and y.
and anti symmetric about x,y=0

$$\mathbf{E} = k(x\hat{\mathbf{i}} - y\hat{\mathbf{j}})$$

But the quad plates are curved.
We assumed in the above that there is no z-dependence.
And that is true only in the limit of $\rho \rightarrow \infty$

Cylindrical coordinates are a better match to our geometry
ρ − radial, z − vertical, θ - azimuthal



Laplacian in cylindrical coordinates

$$\nabla^2 V = \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \cancel{\frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2}} + \frac{\partial^2}{\partial z^2} \right) V$$

Define x $\quad \rho = \rho_0 + x$

The simple symmetric quadratic potential is **not** a solution to this Laplacian

$$V(x,z) \neq \frac{1}{2} k (x^2 - z^2)$$

Cartesian coordinates

$$E_x - iE_y = (b_n - ia_n)\frac{(x+iy)^n}{r_0^n}$$

Satisfies Maxwell for any n

Cartesian coordinates

$$E_x - iE_y = (b_n - ia_n)\frac{(x+iy)^n}{r_0^n}$$

Cylindrical

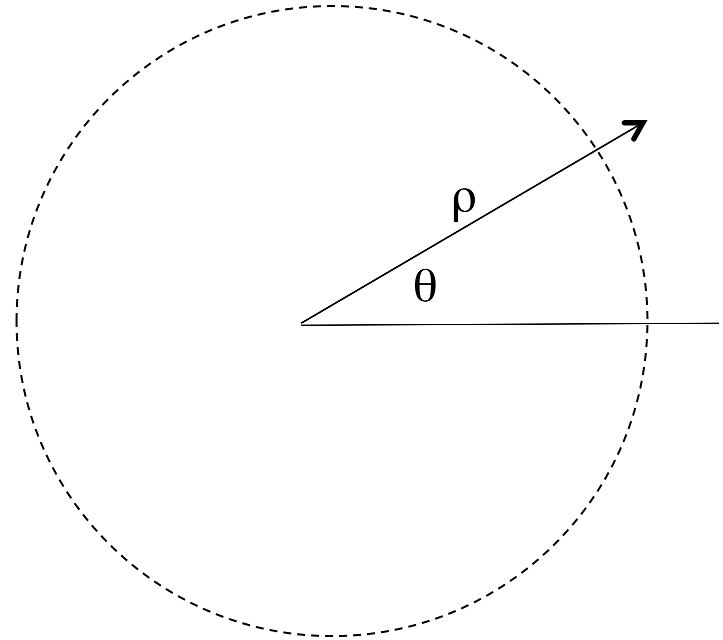
$$\mathbf{E} = -\sum_{n=0}^{\infty} \rho^n \left(a_n \tilde{\nabla}\phi_n^i + b_n \tilde{\nabla}\phi_n^r \right)$$

$$\phi_n^r = \frac{-1}{1+n} \sum_{p=0}^{((n+1)/2)} \binom{n+1}{2p} (-1)^p F_{n+1-2p}(\tilde{r})\tilde{y}^{2p}$$

$$\phi_n^i = \frac{-1}{1+n} \sum_{p=0}^{(n/2)} \binom{n+1}{2p+1} (-1)^p F_{n-2p}(\tilde{r})\tilde{y}^{2p+1}$$

$$\begin{aligned} F_1 &= \ln\tilde{r} \\ F_2 &= \frac{1}{2}(\tilde{r}^2 - 1) - \ln\tilde{r} \\ F_3 &= \frac{3}{2}[-(\tilde{r}^2 - 1) + (\tilde{r}^2 + 1)\ln\tilde{r}] \end{aligned}$$

The simplest (lowest order) solution to the 2 D cylindrical Laplacian is



$$V(\rho,z) = k\left(\frac{1}{2}(\frac{\rho^2}{\rho_0^2}-1) - \ln\frac{\rho}{\rho_0} - \left(\frac{z}{\rho_0}\right)^2\right)$$

Then with the substitution

$$\rho = \rho_0 + x$$

And expanding in the limit where $x \ll \rho_0$

$$\nabla\mathbf{V} = E \sim k\left((x - \frac{x^2}{2\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}}\right)$$

$$\nabla \mathbf{V} = E \sim k\left((x - \frac{x^2}{2\rho_0} + \dots)\hat{\rho} - z\hat{\mathbf{z}} \right)$$

Linear term

Sextupole-like

The solution is not unique.
It is possible to find a solution that is linear in x,
but then it is necessarily nonlinear in z (vertical)

There is inevitably a sextupole component with curved plates independent of the plate shape details and alignment.

Cartesian coordinates − no z- dependence

$$E_x - iE_y = (b_n - ia_n)\frac{(x+iy)^n}{r_0^n}$$

Satisfies Maxwell for any n

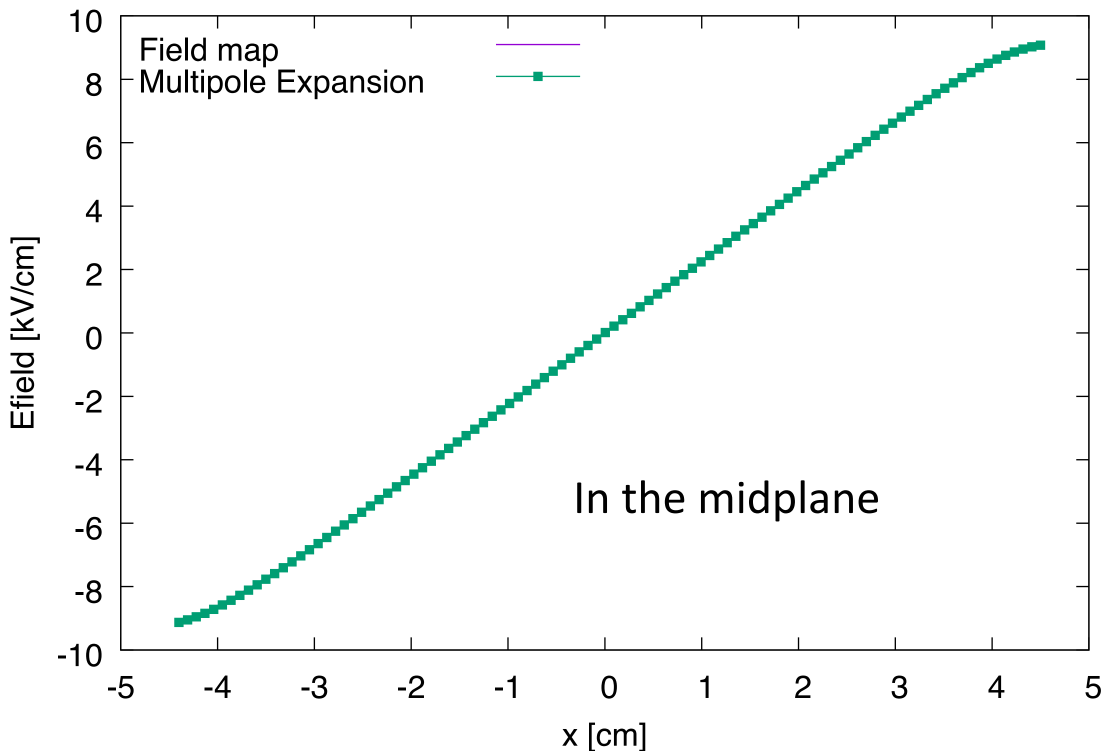
Not so in cylindrical coordinates. But we can expand as

$$E_x = \sum_n b_n \left(\frac{x}{r_0}\right)^n$$

In the midplane (y=0)

$$E_x = \sum_n b_n \left(\frac{x}{r_0}\right)^n$$

Fit to Wanwei's 3-D Opera field map



Multipole	Normal (b_n)	Skew (a_n)
1	1.01609E+06	-1.19899E+00
2	-2.71281E+03	7.71402E+00
3	-1.45524E+04	-1.50954E+01
4	-6.90244E+02	1.46521E+00
5	-5.23865E+03	4.35213E+01
6	1.00671E+02	-4.83666E+01
7	1.21107E+03	7.75800E+01
8	-1.43120E+02	4.99963E+00
9	-9.02621E+04	5.20048E+00
10	2.87638E+02	-3.47144E+01
11	5.36519E+03	3.43053E+01
12	1.05747E+02	2.39598E+00
13	8.00742E+02	-9.14390E+00

This was our guess

$$E \sim k\left((x - \frac{x^2}{2\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}}\right)$$

General form

$$E_x = \sum_n \frac{b_n}{r_0^n} x^n$$

$$\Rightarrow \frac{b_2/r_0^2}{b_1/r_0} = -\frac{1}{2\rho_0} = -0.0703\mathrm{m}^{-1}$$

The fitted values give

$$\Rightarrow \frac{b_2/r_0^2}{b_1/r_0} = -0.0593\mathrm{m}^{-1}$$

$$\nabla V = \mathbf{E} \sim k\left((x - \frac{x^2}{\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}} \right)$$

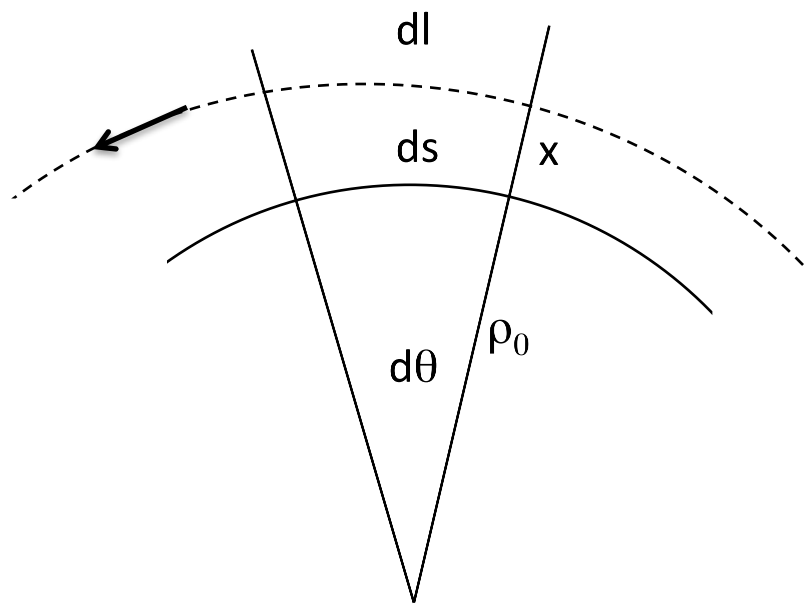
Linear term

Sextupole

The solution is not unique.
It is possible to find a solution that is linear in x,
but then it is necessarily nonlinear in z (vertical)

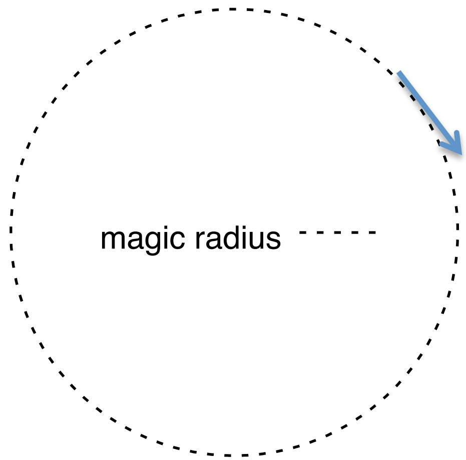
There is inevitably a sextupole component with curved plates independent of the plate shape details and alignment.

Path length

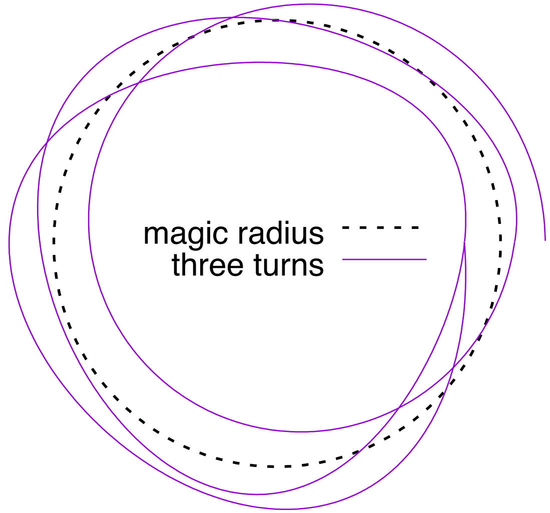


A particle oscillating about the magic radius (ρ_0) spends more time at x>0 than x<0

$$dl = (\rho_0 + x)d\theta$$



The E-field along the trajectory at the magic radius (momentum = p_0) is zero.



But what about the muon with momentum p_0 that oscillates about the magic radius with some betatron amplitude x_β ?

Or the muon with momentum $p_0 + \Delta p$ and betatron amplitude x_β?

$$x = \eta\delta + x_\beta \qquad \delta = \Delta p / p_0$$

Recall

$$\vec{\omega}_a \sim \vec{\omega}_{diff} = -\frac{qe}{m}\left[a_\mu \vec{B} - a_\mu\left(\frac{\gamma}{\gamma+1}\right)(\vec{\beta}\cdot\vec{B})\vec{\beta} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right]$$

Simulation of the effect of E-field or pitch on $\vec{\omega}_a$ at 1ppm would require tracking for 1 million periods or 29 x 10^6 turns!

Instead, to get E-field effect ,compute the average $\langle\vec{\beta}\times\vec{E}\rangle$ along the trajectory

$$\vec{\omega}_a = -\frac{q}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\int\frac{\vec{\beta}\times\vec{E}}{c}\frac{dt}{T}\right]$$

How does E-field correction depend on betatron amplitude?
Consider 4 distinct trajectories,
initialized at t=0 with

$$x' = 0, \Delta p/p = 0.2\%$$

The initial displacements and betatron amplitudes are

1. x = 0.02 m, a_x = 0.738 mm-mrad
2. x = 0.0 m, a_x = 39.7 mm-mrad
3. x = -0.01 m, a_x = 64.3 mm-mrad
4. x= -0.02 m, a_x = 97.66 mm-mrad

Note that the betatron coordinates are related to x and x' according to

$$\begin{aligned} x_\beta &= x - \eta(\Delta p/p) \\ x'_\beta &= x' - \eta'(\Delta p/p) \end{aligned}$$

And $a_x = (\gamma x_\beta^2 + 2\alpha x_\beta x'_\beta + \beta {x'_\beta}^2)^{1/2}$ is the betatron amplitude

E-field correction

$$C_e = \left(1 - \frac{1}{a_\mu}\frac{m^2}{p^2}\right)\frac{\beta E_r}{cB} \sim \left(1 - \frac{m^2}{a_\mu p_0^2}(1 - 2\frac{\Delta p}{p})\right)\frac{\beta E_r}{cB} \qquad (1)$$

Magic momentum $\qquad m^2/p_0^2 = a_\mu$

$$x_e = \eta\delta$$

$$C_e(\delta, x_{\beta 0}) \quad \approx \quad 2\frac{\beta k}{cB}\left(\frac{x_e^2}{\eta} + \frac{1}{2\rho_0}(\frac{x_e^3}{\eta} + \frac{1}{2}x_{\beta 0}^2\frac{x_e}{\eta})\right)$$

If $\langle x_e\rangle = \langle\delta\rangle\eta = 0$ then correction is independent of x_β

$$\langle C_e(\delta, x_{\beta 0})\rangle \quad = \quad 2\frac{\beta k}{cB}\eta\langle\delta^2\rangle$$

If $\langle x_e \rangle = \langle \delta \rangle \eta \neq 0$ then according to the Miller/Nguyen rule

To minimize the E-field correction choose p_0 so that

$$2a_\mu \langle \frac{p - p_0}{p_0} \rangle = \frac{m^2}{p_0^2} - a_\mu$$

Then

$$\langle C_e \rangle \sim 2 \left[-\eta(\langle \delta \rangle^2 - \langle \delta^2 \rangle) + \frac{1}{4\rho_0}(\langle \delta \rangle^2 \langle x_{\beta 0}^2 \rangle \right] \frac{\beta k}{cB}$$

Contribution from betatron amplitude

Correction *increases* with betatron amplitude