Simulation of E-field and Pitch systematics and dependence on betatron amplitude

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E-field contribution

$$\vec{\omega}_a = -\frac{q}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{m^2}{p^2}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right]$$

Averaging over the distribution, assuming E-field is linear in displacement, continuous quad approximation, independent of betatron amplitude . . .

$$2n(n-1)\beta^2\langle x_e^2\rangle/R_0^2$$

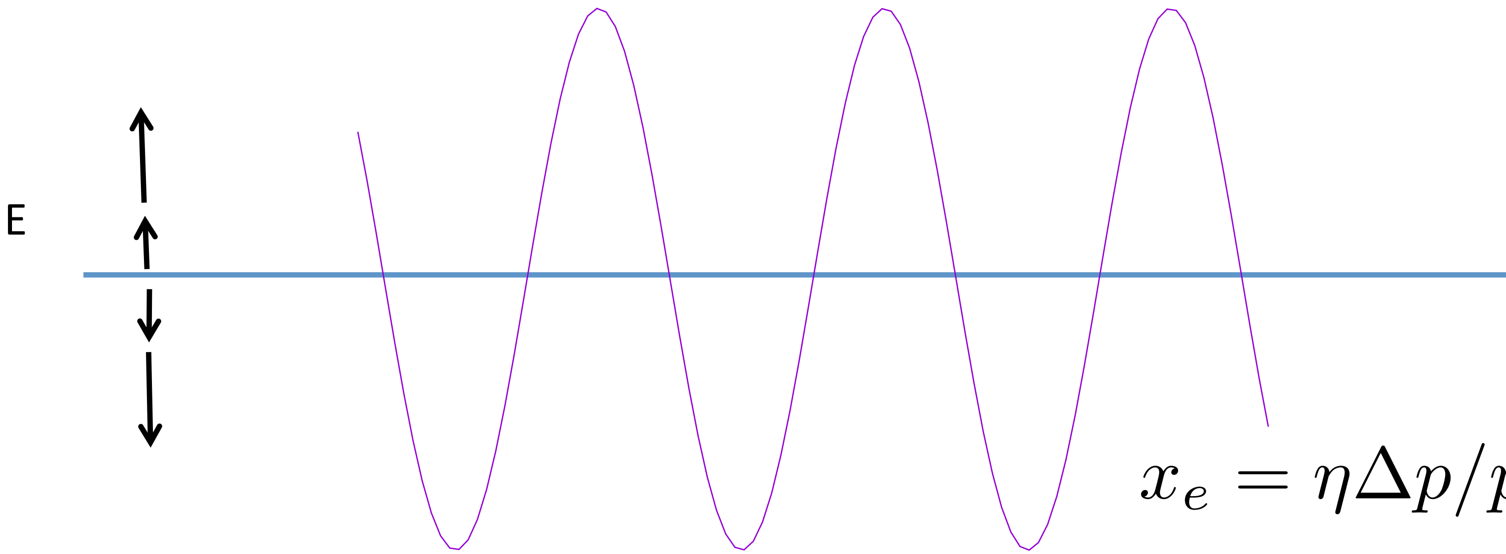
where x_e is the equilibrium radial displacement for particle with momentum $\Delta p/p$

Let's check those assumptions

$$\vec{\omega}_a = -\frac{q}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{m^2}{p^2}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right]$$

E-field contribution
Quad linearity

In an ideal cartesian geometry and quadrupole where the horizontal field is antisymmetric about the closed orbit, the E-field correction is independent of betatron amplitude



$$x_e = \eta \Delta p/p$$

In a curved geometry, the integrated E-field along the trajectory depends on betatron amplitude in two ways

1. Sextupole (quadratic) component of the quads
   - Sextupole component is symmetric about magic radius
   - Shifts the 'closed orbit'
2. Path length (asymmetric about magic radius)

$$\nabla V = \mathbf{E} \sim k\left((x - \frac{x^2}{\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}}\right)$$

$$x = \eta\delta + x_\beta$$

$$\begin{aligned}
\langle E_r(s)\rangle &= k\langle\left(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2\right)\rangle \\
&= \frac{k}{L}\int_0 \left(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2\right) dl \\
&= \frac{k}{L}\int_0 \left(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2\right)(1 + x_\beta/\rho_0)ds
\end{aligned}$$

sextupole

Path length

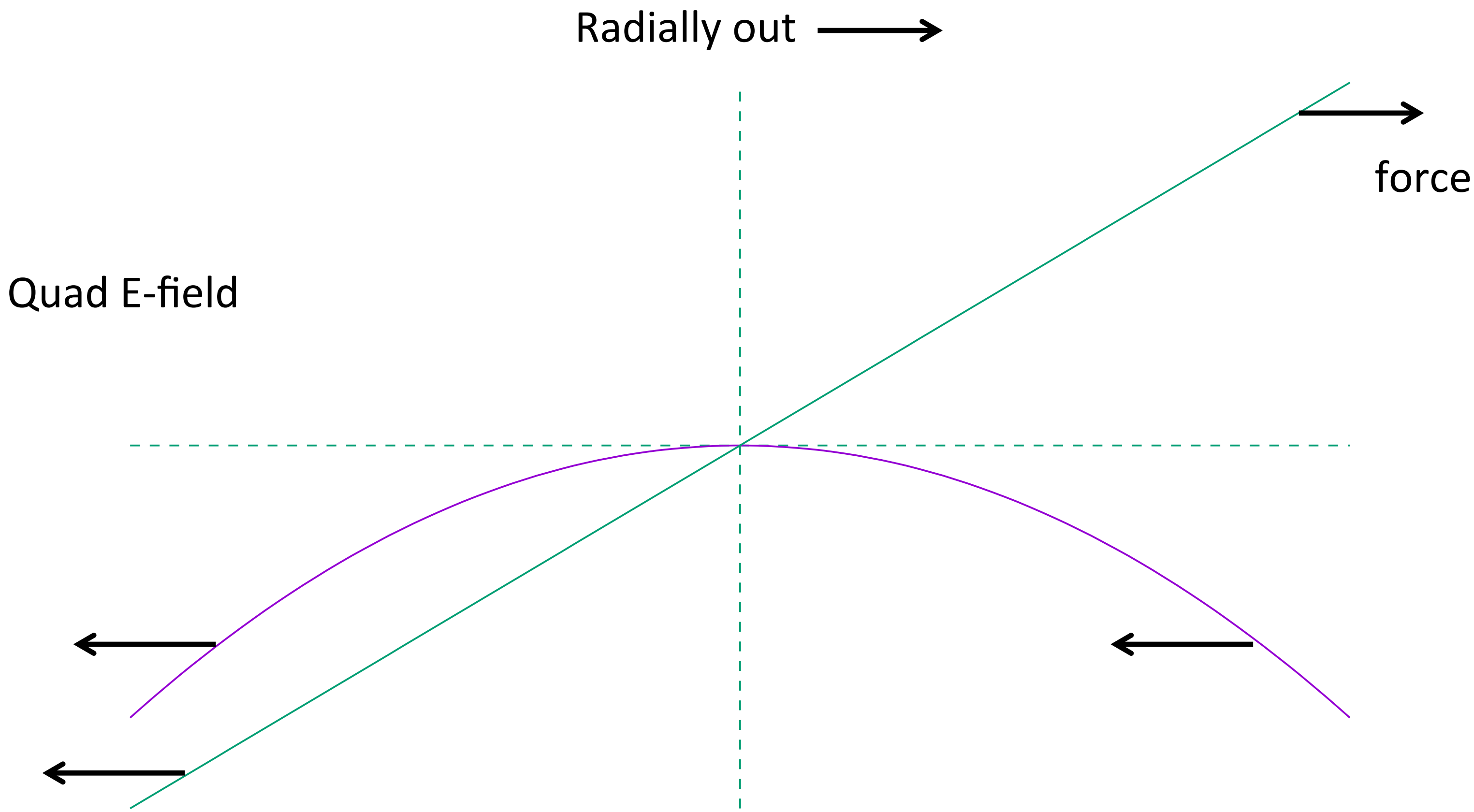
The average E-field for a muon with momentum $p_0 + \Delta p$ and betatron amplitude x_β is

$$\langle E_r \rangle \quad = \quad k \left(\eta\delta + \frac{1}{2\rho_0}((\eta\delta)^2 + \frac{1}{2}x_{\beta 0}^2) + \ldots \right)$$

$$(x_e = \eta\delta)$$

For positive momentum offset correction *increases* with betatron amplitude

Closed orbit depends on betatron amplitude



Sextupole component shifts <x> radially inward by

$$\langle F_x \rangle \sim k \frac{x_\beta^2}{2\rho}$$

E-field contribution along any trajectory:

$$\vec{C}_e(T) \sim 2\frac{\Delta p}{p}\frac{1}{T}\int_0^T \frac{\vec{\beta}\times\vec{E}}{Bc}dt$$

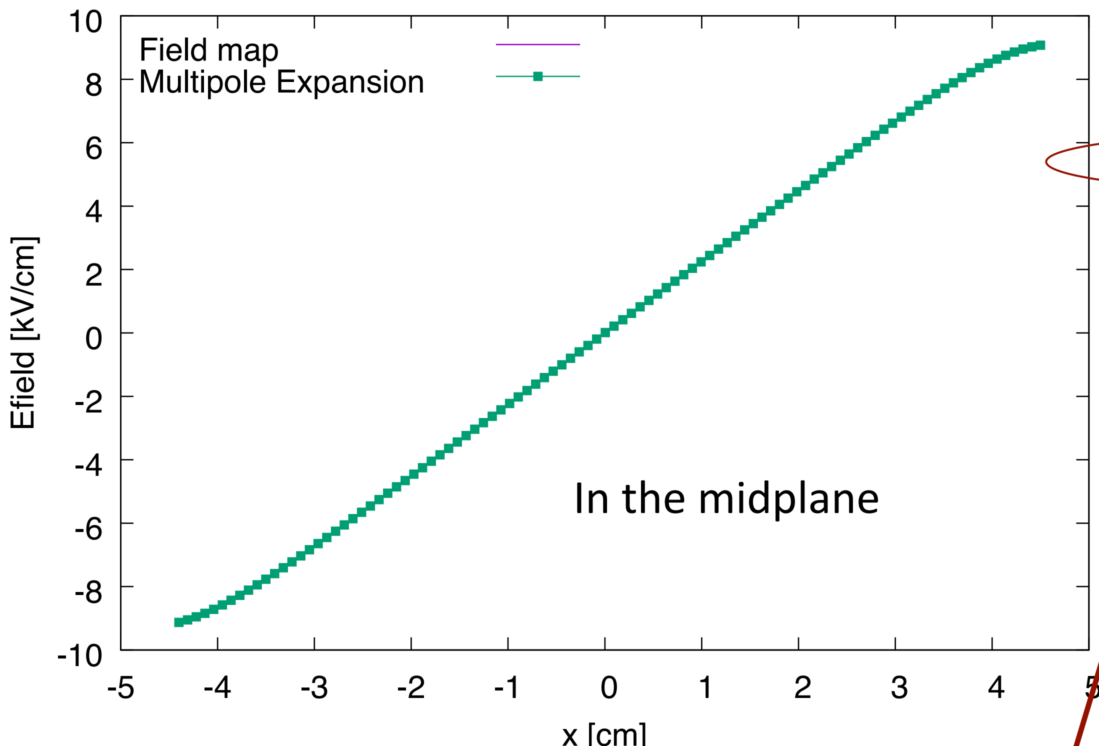
To compute correction in simulation, integrate $\langle\vec{\beta}\times\vec{E}\rangle$ along the trajectory of the muon

=> E-field correction as a function of time, $C_e(t)$

$$E_x = \sum_n b_n \left(\frac{x}{r_0} \right)^n$$

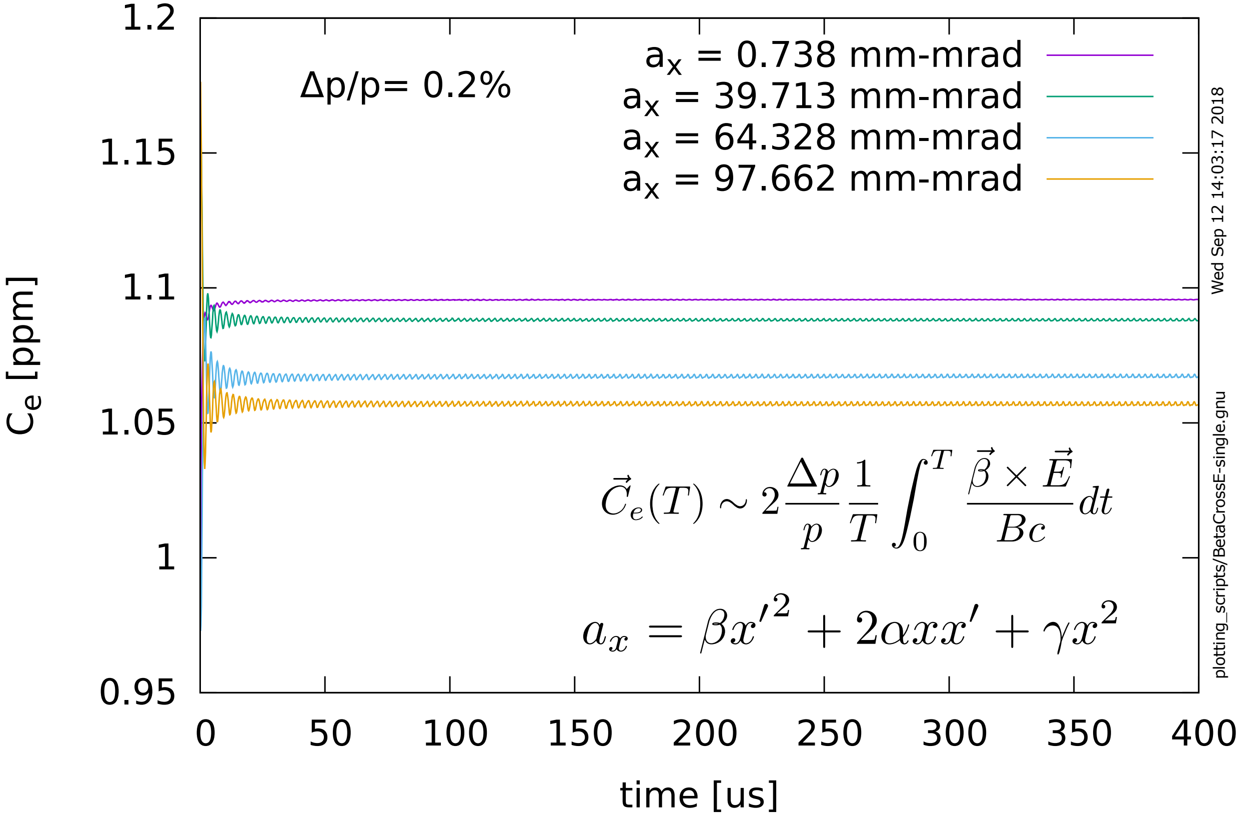
Quad multipoles

Fit to Wanwei's 3-D Opera field map



Multipole	Normal (b_n)	Skew (a_n)
1	1.01609E+06	-1.19899E+00
2	-2.71281E+03	7.71402E+00
3	-1.45524E+04	-1.50954E+01
4	-6.90244E+02	1.46521E+00
5	-5.23865E+03	4.35213E+01
6	1.00671E+02	-4.83666E+01
7	1.21107E+03	7.75800E+01
8	-1.43120E+02	4.99963E+00
9	-9.02621E+04	5.20048E+00
10	2.87638E+02	-3.47144E+01
11	5.36519E+03	3.43053E+01
12	1.05747E+02	2.39598E+00
13	8.00742E+02	-9.14390E+00

Sextupole component



- C_e(t) oscillates with betatron frequency at early time
- C_e(t) decreases with increasing betatron amplitude
- Contribution from betatron amplitude < 40 ppb

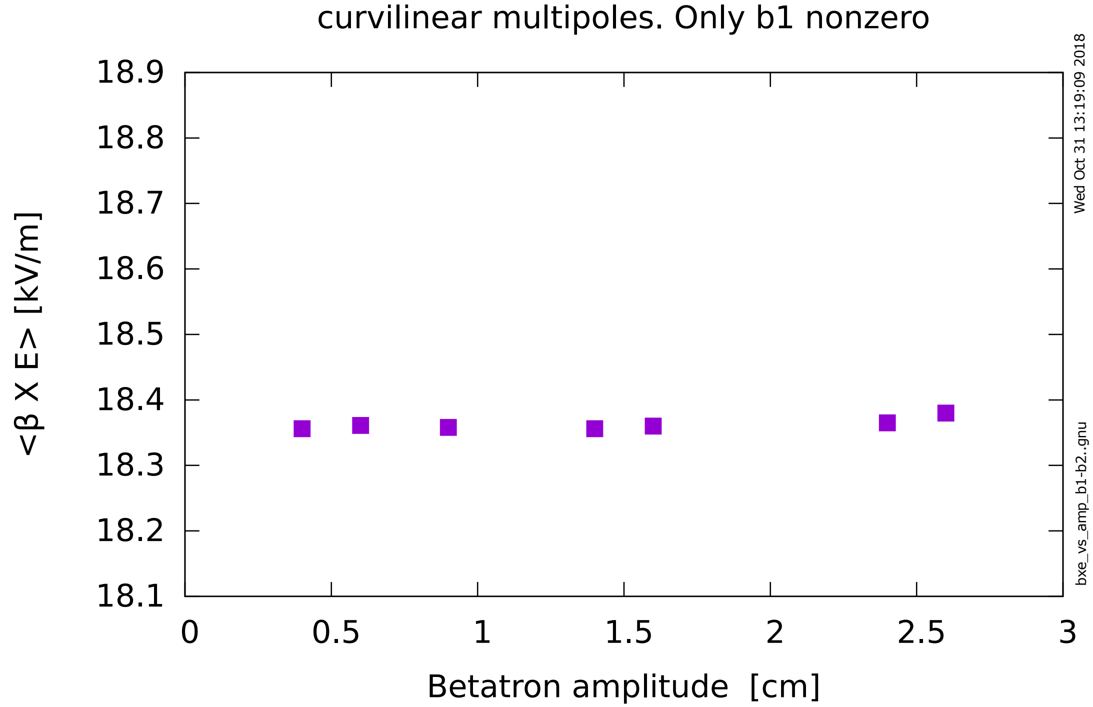
Correction *decreases* with betatron amplitude

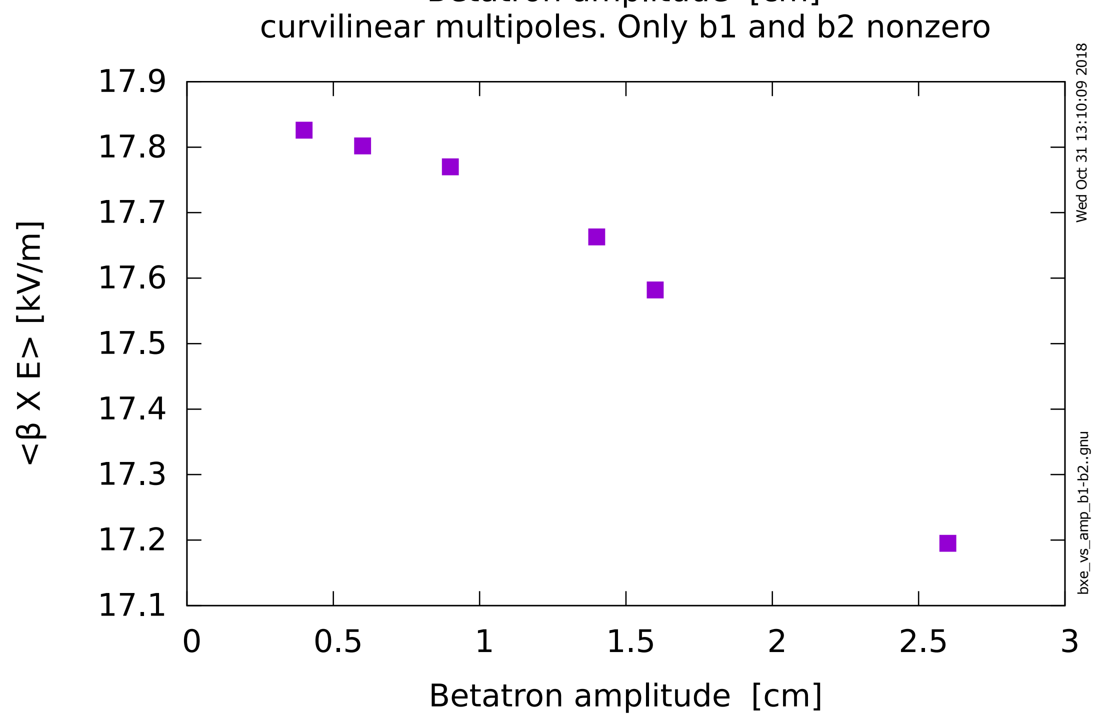
Dependence of E-field correction on Betatron amplitude with and without sextupole-like

$$\Delta p/p = 0.2\%$$

Efield correction decreases with betatron amplitude for $\Delta p/p > 0$

And increases for $\Delta p/p < 0$





What do we know so far?

If $\Delta p/p = 0.2\%$, then due to the quadratic dependence of E-field on displacement

- E-Field contribution decreases by ~ 3.5% as betatron amplitude increases from 0 to 2.5 cm

(The quadratic dependence is evident in 3-D Opera quadrupole field map)

Does it really matter?

E-field contribution for each muon

$$\vec{C}_e(T) \sim 2\frac{\Delta p}{p}\frac{1}{T}\int_0^T \frac{\vec{\beta}\times\vec{E}}{Bc}dt$$

Averaging over the distribution assuming E-field is linear in displacement

$$2n(n-1)\beta^2\langle x_e^2\rangle/R_0^2$$

where x_e is the equilibrium radial displacement for particle with momentum $\Delta p/p$

$$x_e = \eta\Delta p/p$$

Let's check to see if the linear formula gives the correct answer for a realistic distribution

Characterization of a simulated distribution

- Propagate a 'realistic' distribution through the injection channel (through backleg iron and inflector and into storage ring)
  - Assume longitudinal distribution is as measured in Spring 18
  - Kicker pulse shape − as measured
- Track around the ring − (quadrupole field as per Opera 3D map) until muon decays
- For each muon that decays at t > 35μs record:
  - Momentum
  - End phase space coordinates, decay time,
    - closed orbit

$$x_e = \frac{1}{N}\sum_{i=1}^{N} x_i$$

    - Fast rotation frequency

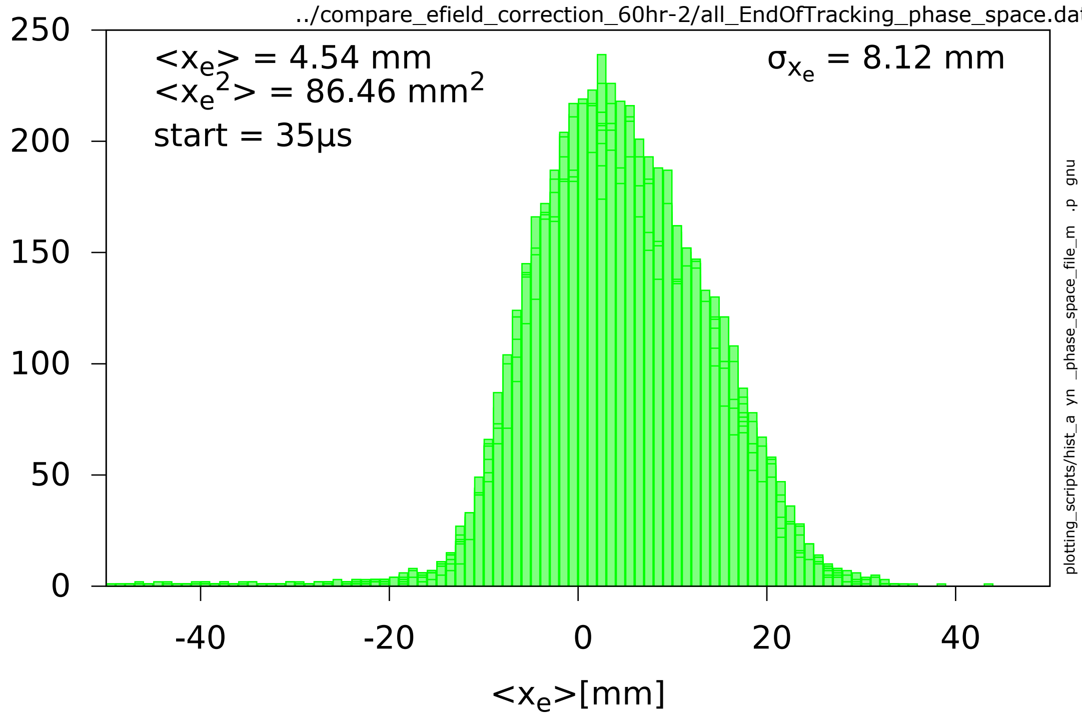
$$f_{FR} = \frac{N}{T}$$

    - Efield contribution

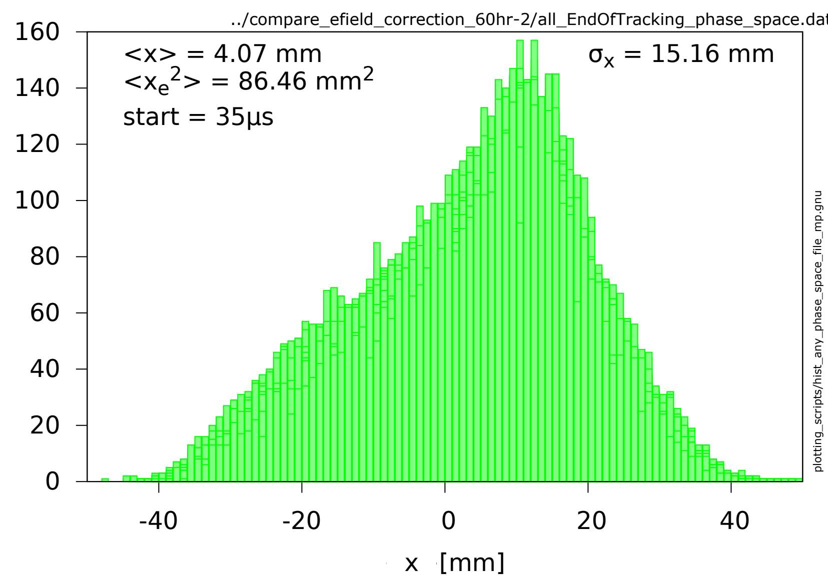
$$\vec{C}_e(T) \sim 2\frac{\Delta p}{p}\frac{1}{T}\int_0^T \frac{\vec{\beta}\times\vec{E}}{Bc}dt$$

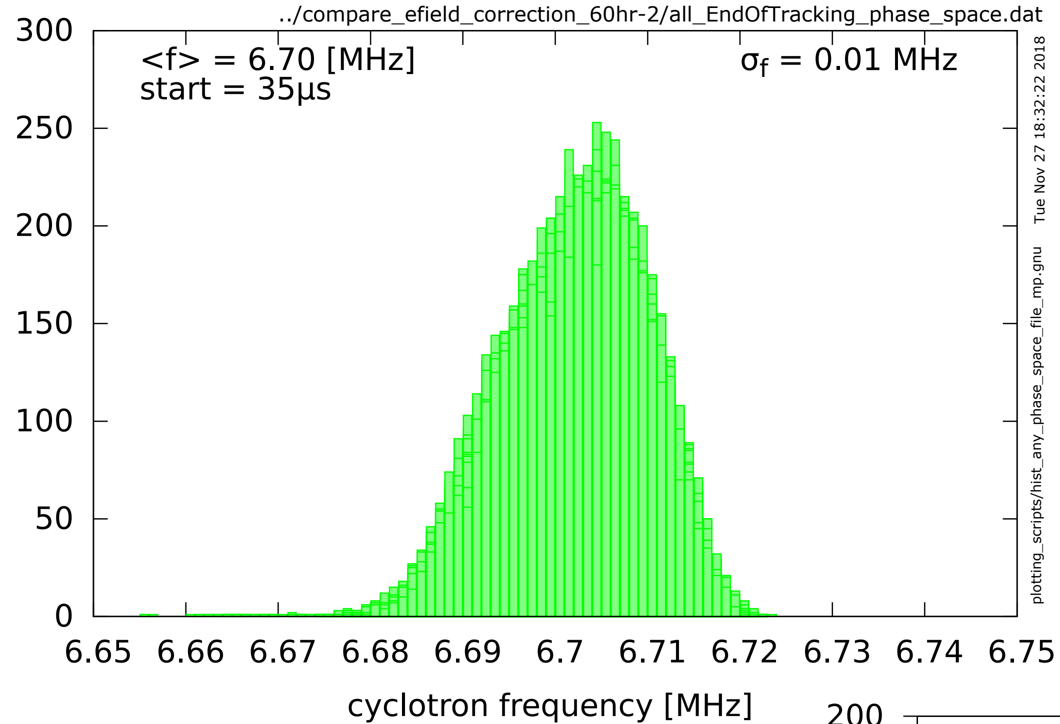
Quad 18.3/13.1 kV
Kicker 204 G (~70%)

Distribution of equilibrium radii



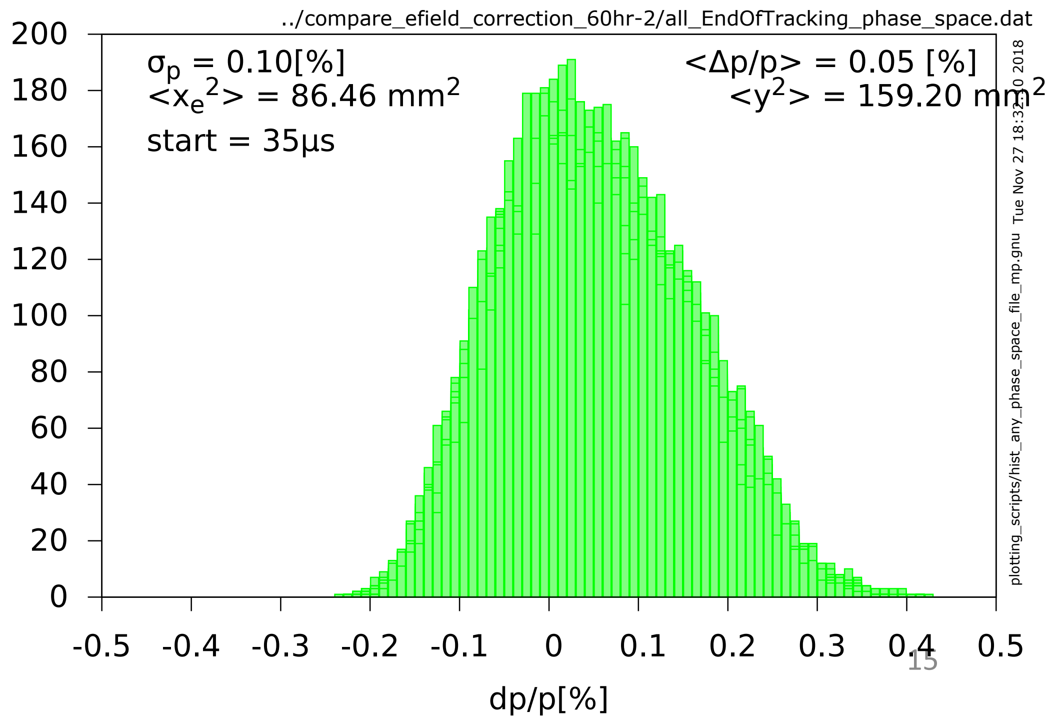
X at decay point

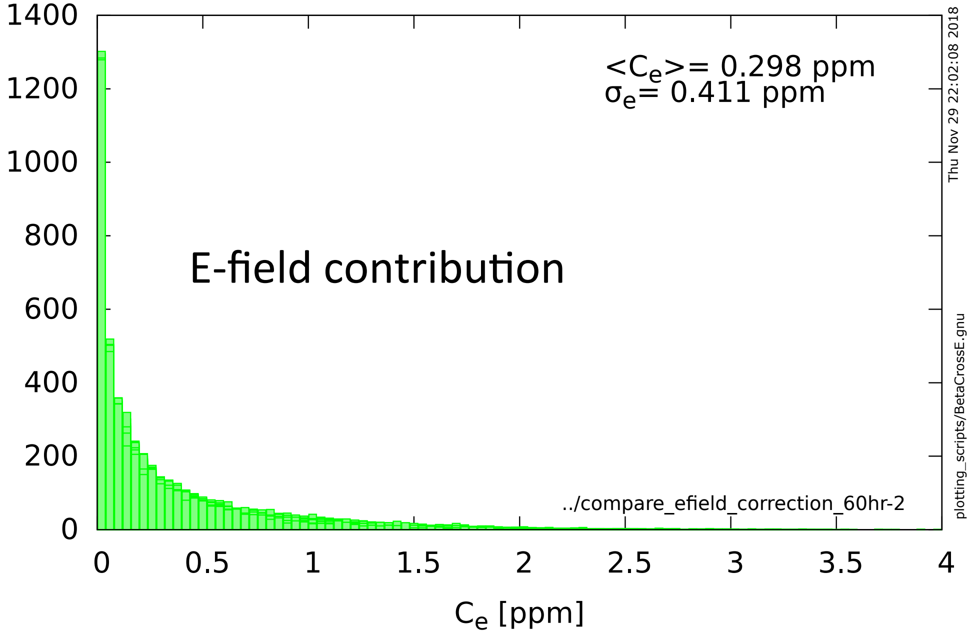


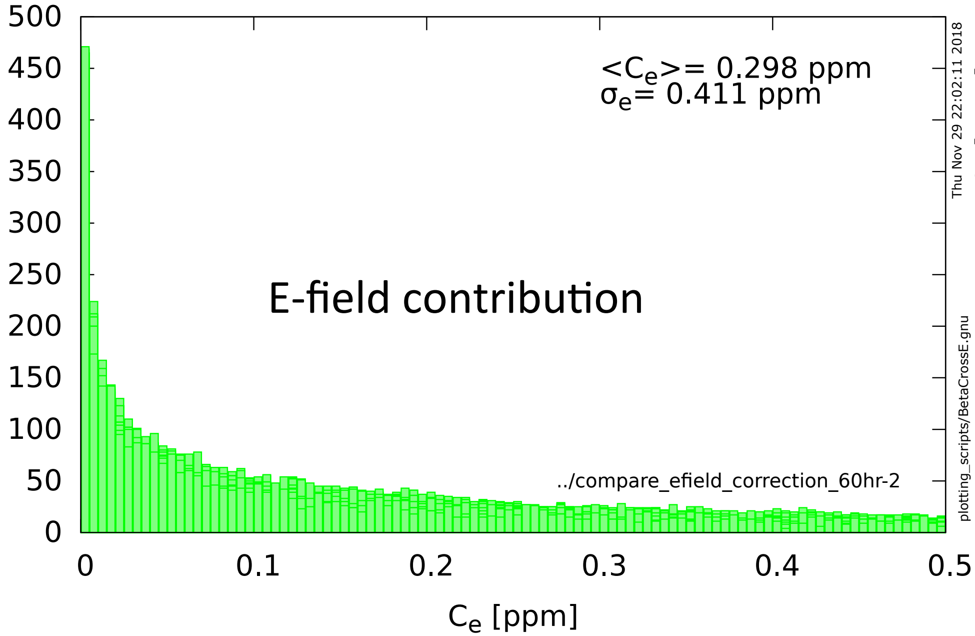


Fast rotation
frequency distribution

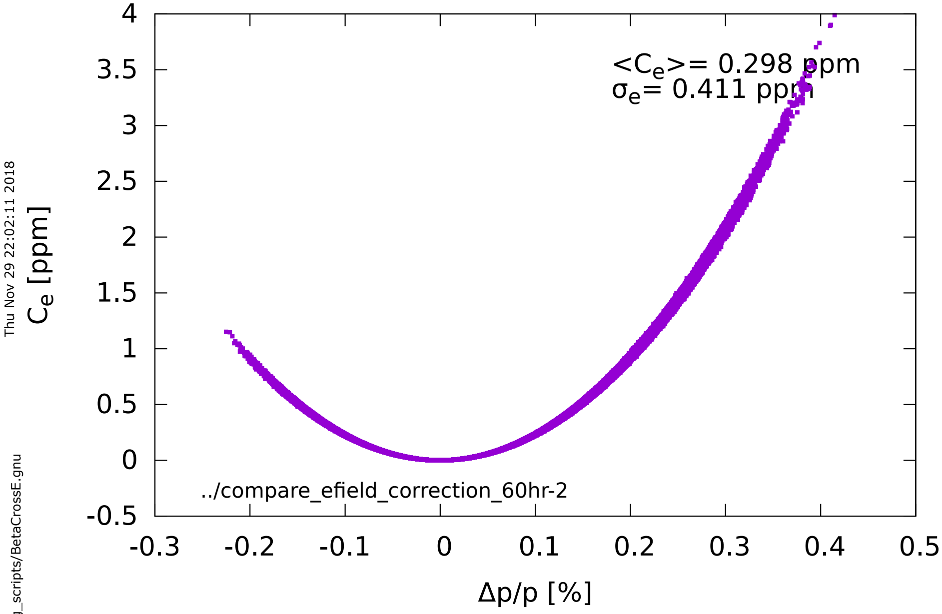
Momentum distribution







E-field conribution vs momentum



Next step. Since we have C_e and x_e for every muon we can compare

Exact $$\langle C_e \rangle = 2\langle \frac{\Delta p}{p} \frac{1}{T} \int_0^T \frac{\vec{\beta} \times \vec{E}}{Bc} dt \rangle$$ = 0.298 ppm

to

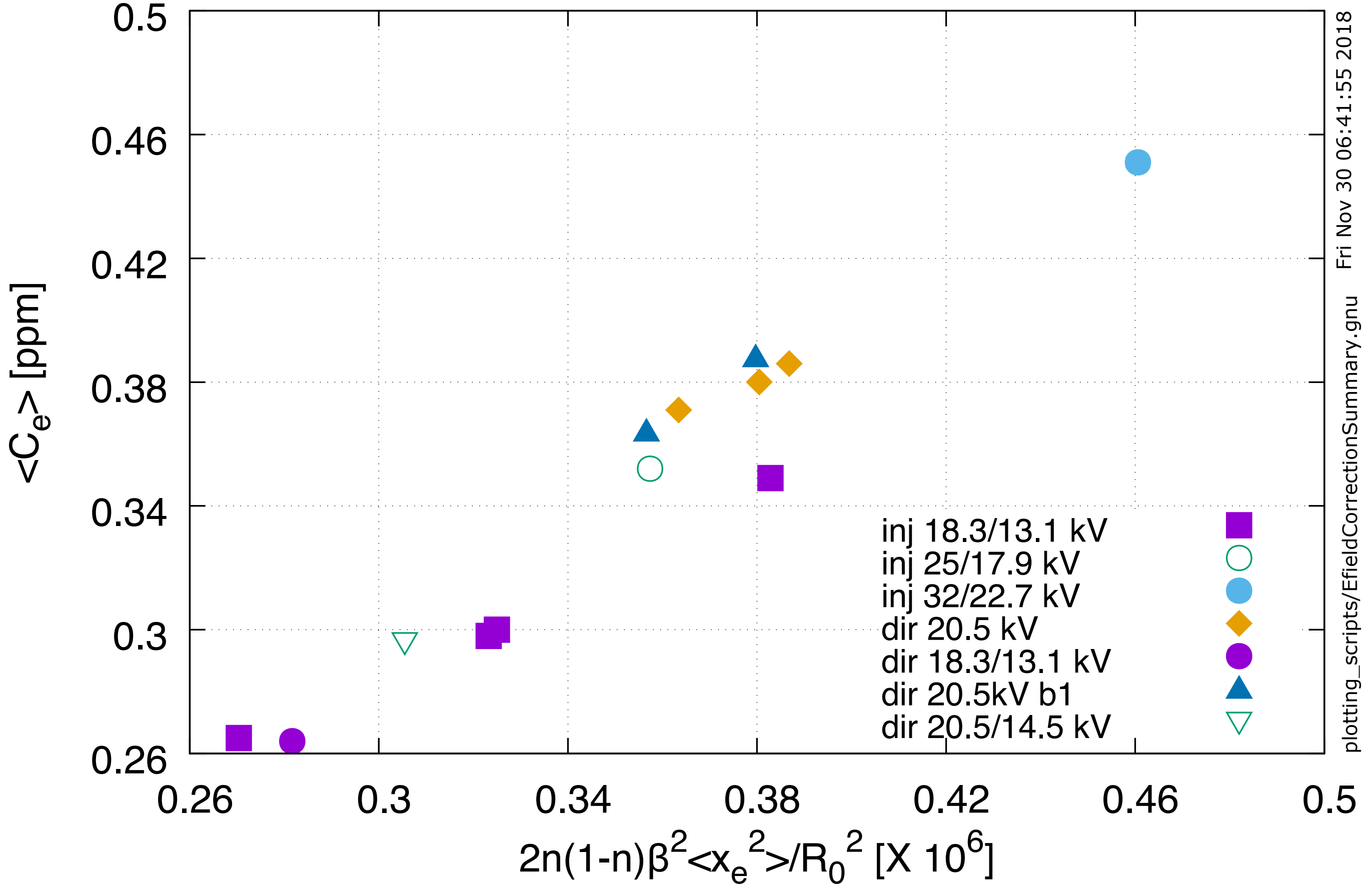
Linear approximation $$2n(n-1)\beta^2 \langle x_e^2 \rangle / R_0^2$$ = 0.323 ppm

Conclusion:

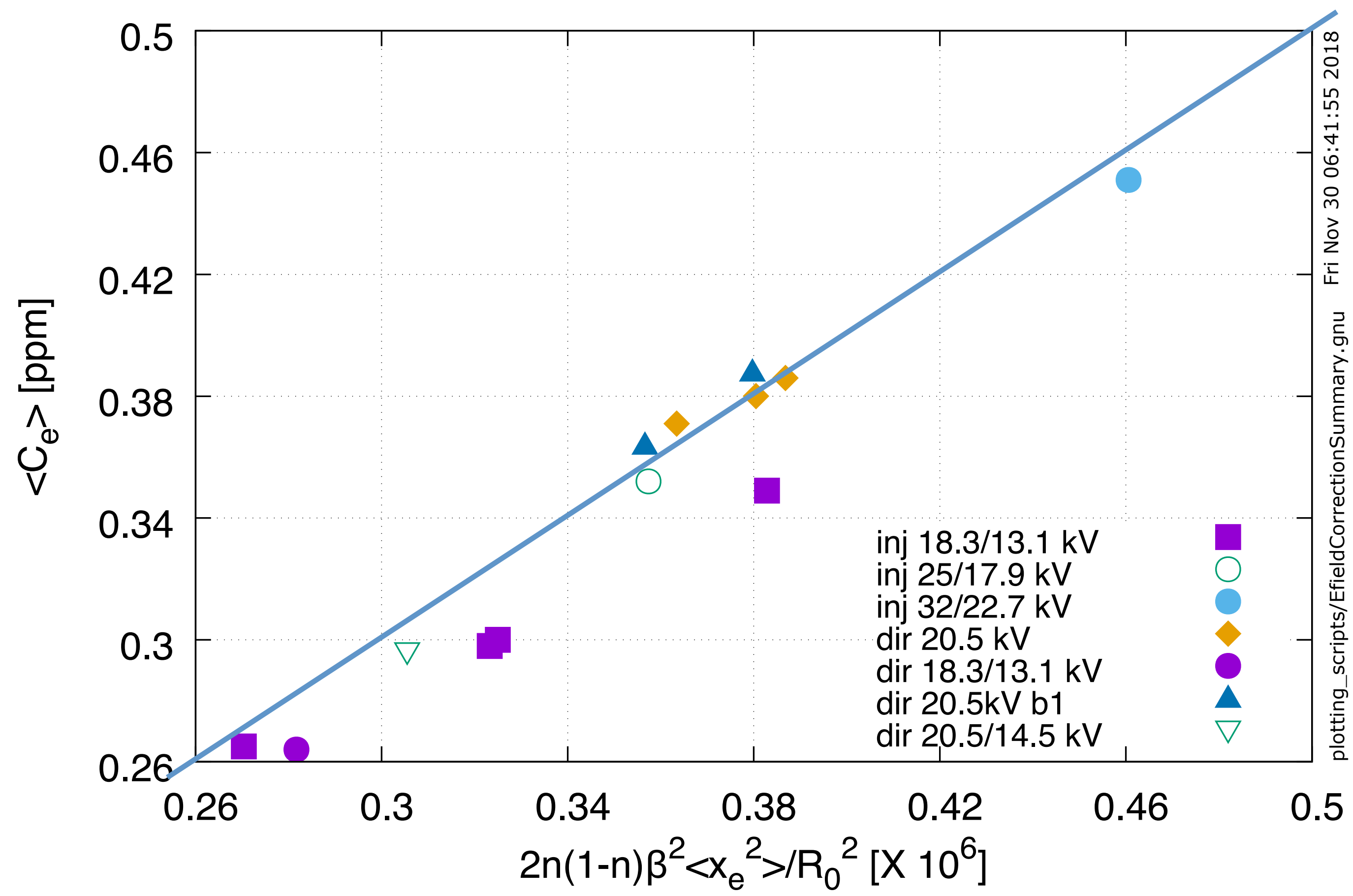
If we reconstruct the distribution of equilbrium radii *perfectly*, there is an 8% discrepancy (25 ppb)

How does correlation depend on kicker, quads, scraping ?

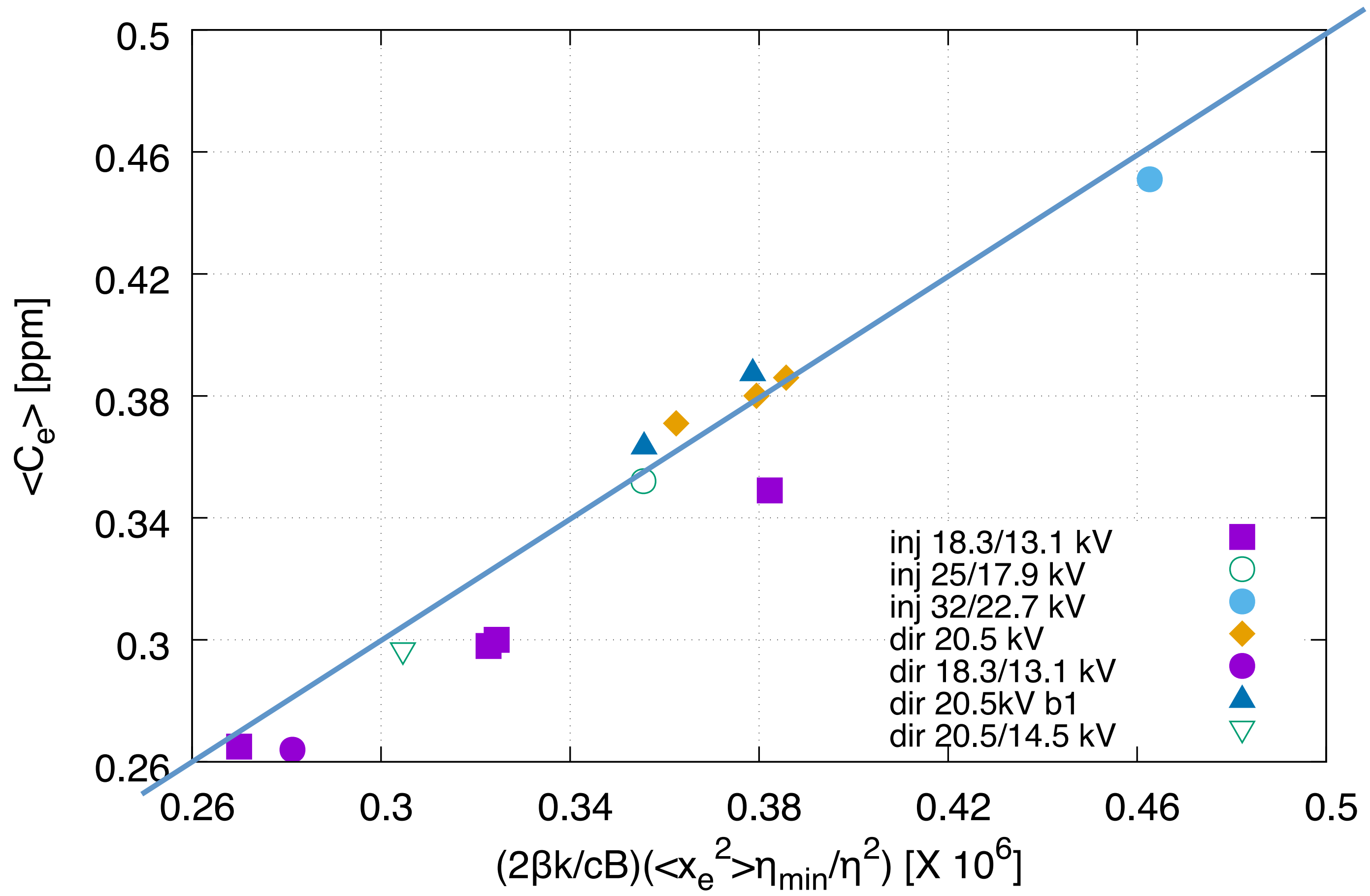
Config	Quad [kV]	kicker [G]	scraping	Inject
1	20.5	-	no	On axis
2	20.5	-	no	On axis – dx = 1cm
3	20.5	-	no	On axis – dp/p=0.1%
4	18.3/13.1	-	yes	On axis – dp/p=0.1%
5	18.3/13.1	204	yes	Through inflector
6	18.3/13.1	175	yes	Through inflector
7	18.3/13.1	172/267/183	yes	Through inflector
8	18.3/13.1	292	yes	Through inflector
9	25./17.9	292	yes	Through inflector
10	32./22.7	292	yes	Through inflector
11	20.5	-	yes	On axis – linear quad field
12	20.5	-	yes	On axis – linear quad – dp/p=0.1%
13	20.5/14.5	-	Yes	On axis



$\langle C_e \rangle$ [ppm]
0.5
0.46
0.42
0.38
0.34
0.3
0.26
0.26
0.3
0.34
0.38
0.42
0.46
0.5
$2n(1-n)\beta^2\langle x_e^2\rangle/R_0^2$ [X 10^6]
inj 18.3/13.1 kV
inj 25/17.9 kV
inj 32/22.7 kV
dir 20.5 kV
dir 18.3/13.1 kV
dir 20.5kV b1
dir 20.5/14.5 kV
Fri Nov 30 06:41:55 2018
plotting_scripts/EfieldCorrectionSummary.gnu



0.5
0.46
0.42
0.38
0.34
0.3
0.26
0.26
0.3
0.34
0.38
0.42
0.46
0.5
$<C_e>$ [ppm]
$2n(1-n)\beta^2<x_e{}^2>/R_0{}^2$ [X 10^6]
inj 18.3/13.1 kV
inj 25/17.9 kV
inj 32/22.7 kV
dir 20.5 kV
dir 18.3/13.1 kV
dir 20.5kV b1
dir 20.5/14.5 kV
Fri Nov 30 06:41:55 2018
plotting_scripts/EfieldCorrectionSummary.gnu



0.5
0.46
0.42
0.38
0.34
0.3
0.26
<Ce> [ppm]
0.26
0.3
0.34
0.38
0.42
0.46
0.5
(2βk/cB)(<xe2>ηmin/η2) [X 106]
inj 18.3/13.1 kV
inj 25/17.9 kV
inj 32/22.7 kV
dir 20.5 kV
dir 18.3/13.1 kV
dir 20.5kV b1
dir 20.5/14.5 kV

We find that

If we measure equilibrium radial distribution perfectly then we know E field contribution to better than 10%

Check pitch correction

Pitch

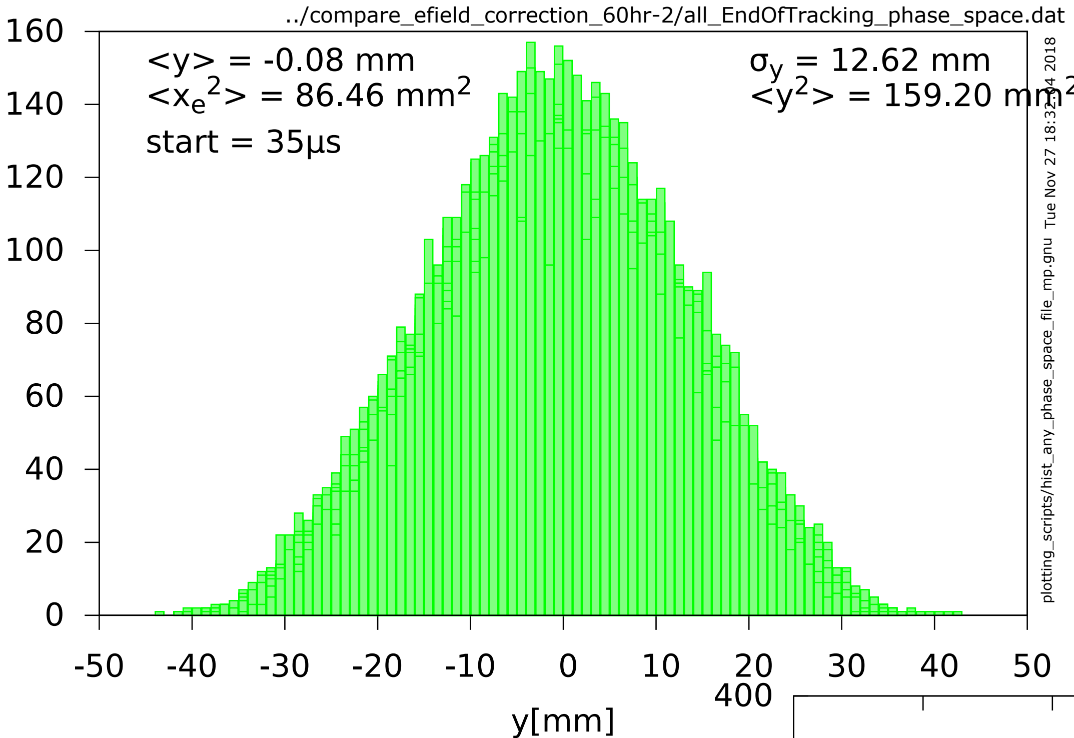
$$\vec{\omega}_a \sim \vec{\omega}_{diff} = -\frac{qe}{m}\left[a_\mu \vec{B} - a_\mu \left(\frac{\gamma}{\gamma+1}\right)(\vec{\beta}\cdot\vec{B})\vec{\beta} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right]$$

Exact

$$\vec{C}_p(T) = \frac{1}{B}\frac{1}{T}\int_0^T (\vec{\beta}\cdot\vec{B})\vec{\beta}dt$$

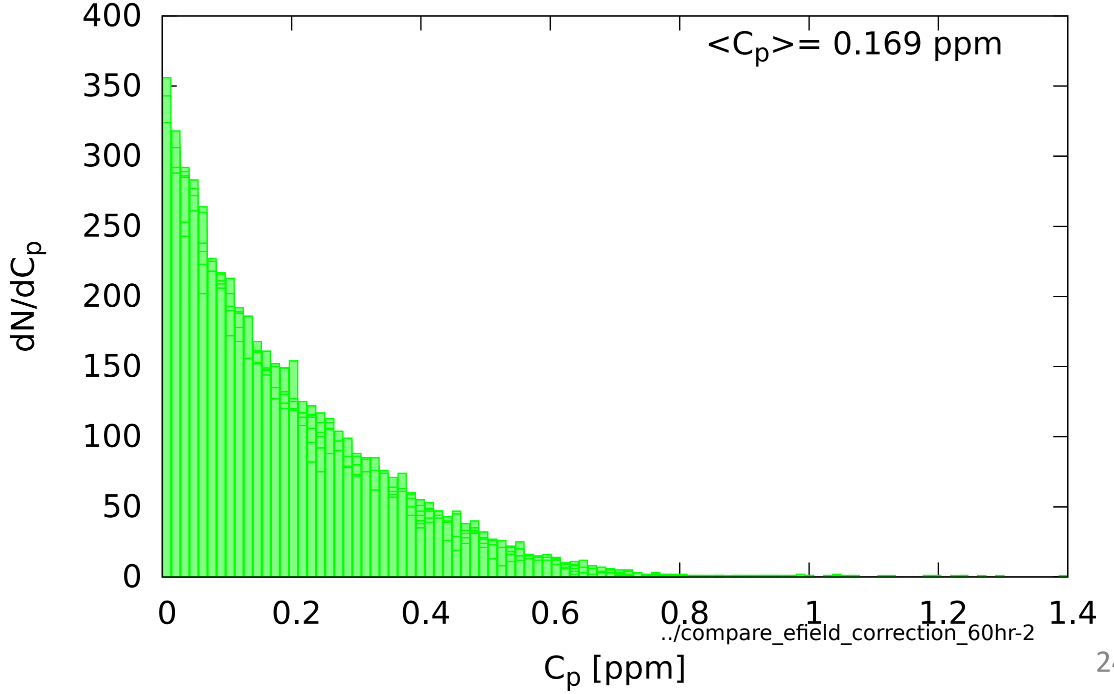
First order

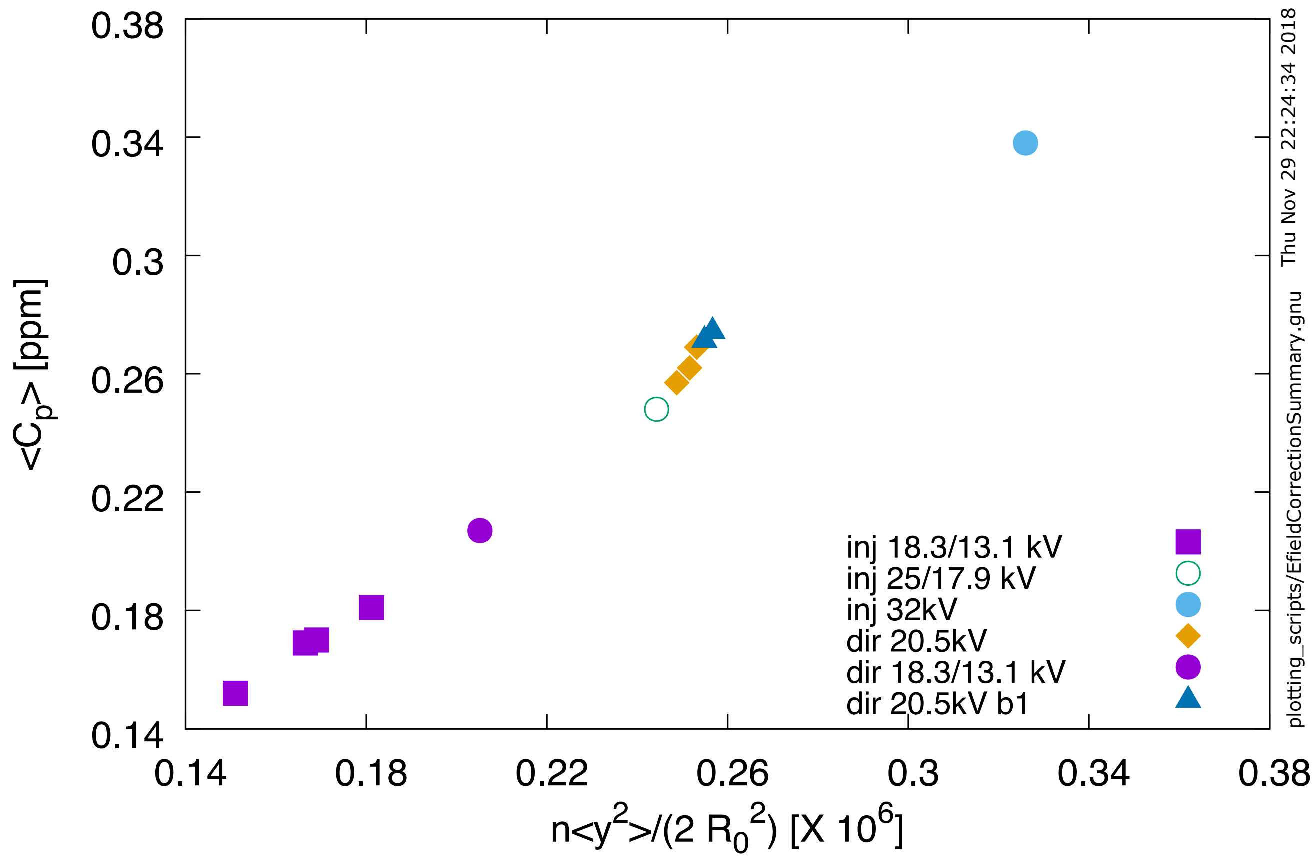
$$\langle C_p \rangle \sim \frac{n\langle y^2 \rangle}{R_0^2}$$



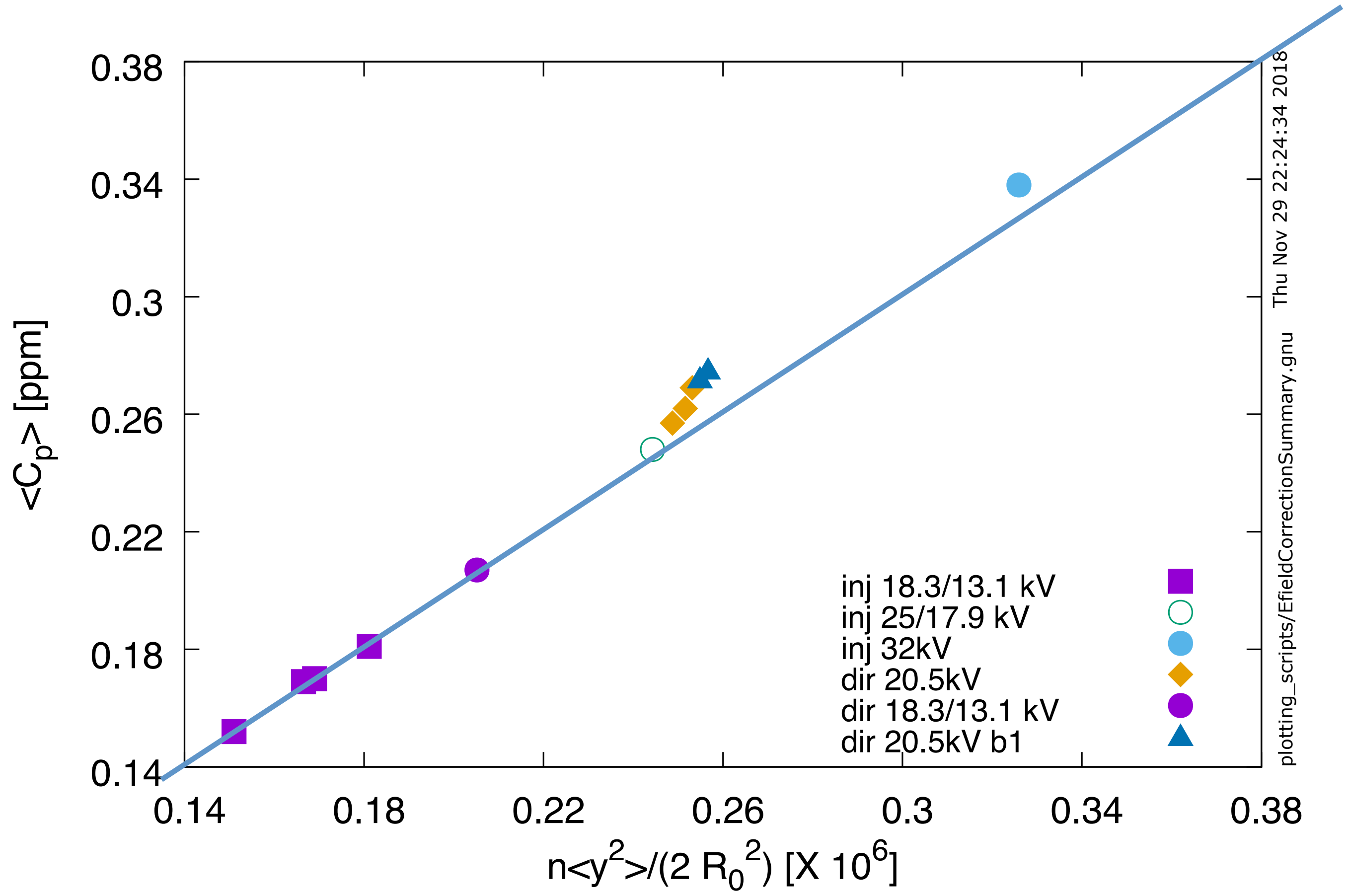
Vertical distribution of decay muons

Distribution of pitch contribution





0.38
0.34
0.3
0.26
0.22
0.18
0.14
<Cp> [ppm]
0.14
0.18
0.22
0.26
0.3
0.34
0.38
n<y2>/(2 R02) [X 106]
inj 18.3/13.1 kV
inj 25/17.9 kV
inj 32kV
dir 20.5kV
dir 18.3/13.1 kV
dir 20.5kV b1
Thu Nov 29 22:24:34 2018
plotting_scripts/EfieldCorrectionSummary.gnu



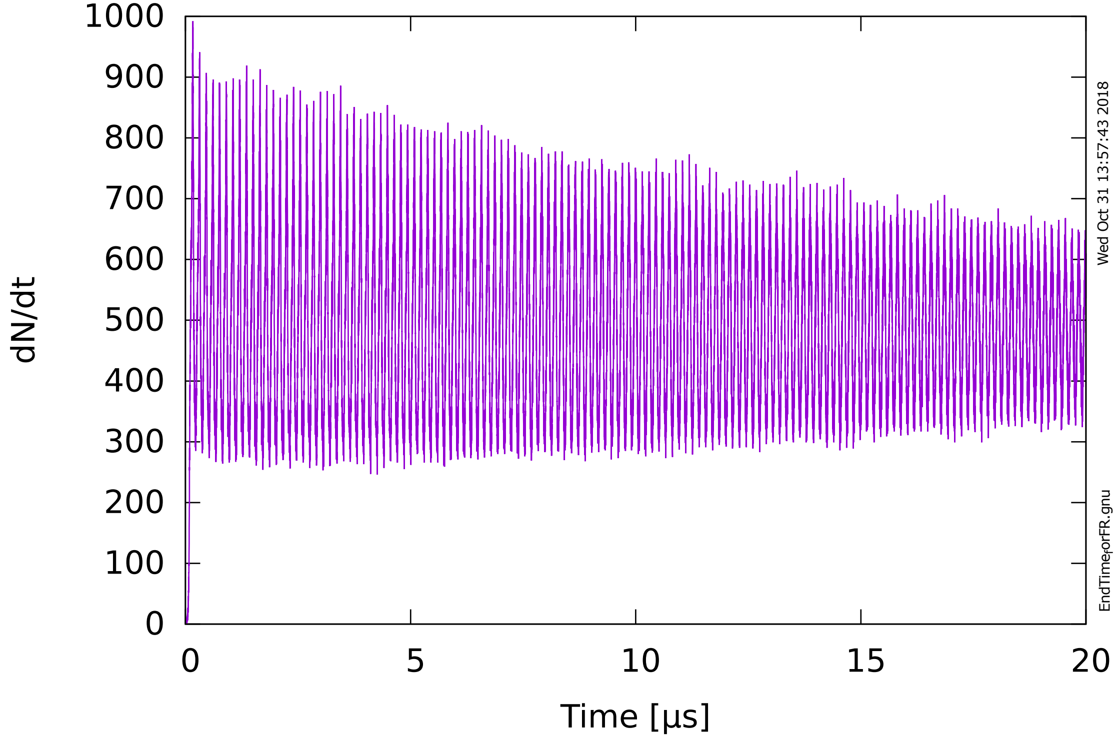
0.38
0.34
0.3
0.26
0.22
0.18
0.14
<C_p> [ppm]
0.14
0.18
0.22
0.26
0.3
0.34
0.38
n<y^2>/(2 R_0^2) [X 10^6]
inj 18.3/13.1 kV
inj 25/17.9 kV
inj 32kV
dir 20.5kV
dir 18.3/13.1 kV
dir 20.5kV b1
Thu Nov 29 22:24:34 2018
plotting_scripts/EfieldCorrectionSummary.gnu

And

If we measure vertical distribution perfectly
then we know pitch contribution to 5%

Next. Check how well the fast rotation analysis reproduces radial distribution?

Simulation also generates a fast rotation signal, namely the distribution of muon decay times



Fast Rotation Signal

Summary

- Effect of E-field and pitch is computed in tracking simulation by integrating (summing along the trajectory)

$$\vec{C}_e(T) \sim 2\frac{\Delta p}{p}\frac{1}{T}\int_0^T \frac{\vec{\beta}\times\vec{E}}{Bc}dt$$

$$\vec{C}_p(T) = \frac{1}{B}\frac{1}{T}\int_0^T (\vec{\beta}\cdot\vec{B})\vec{\beta}dt$$

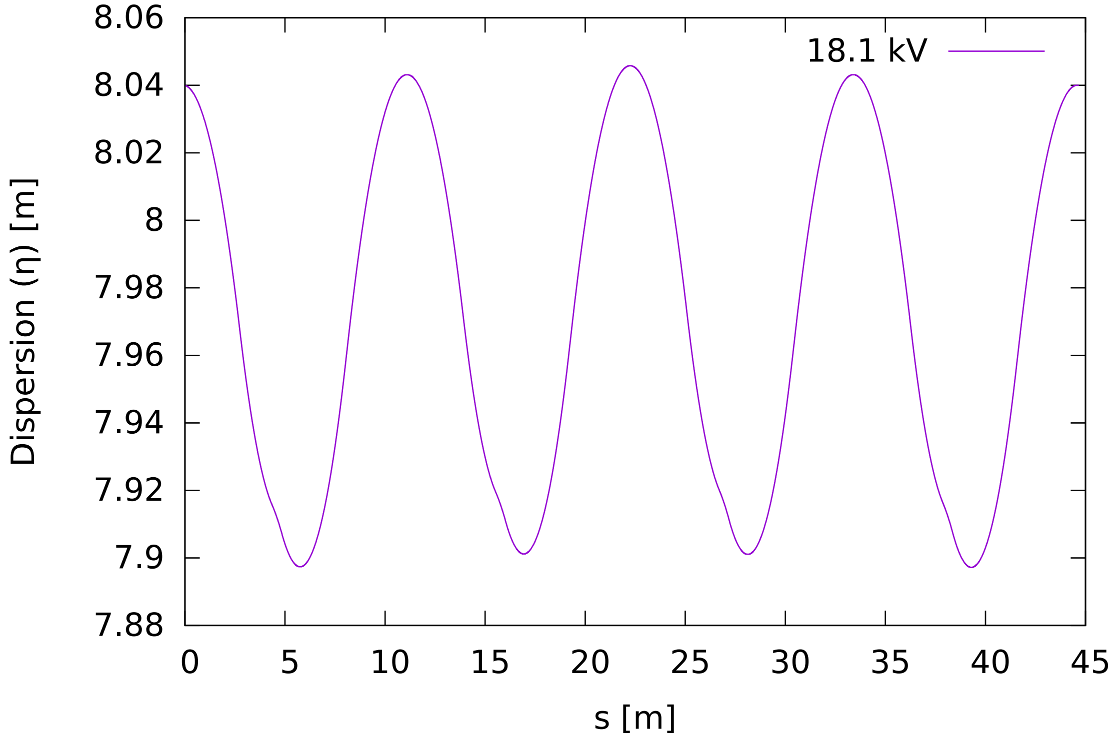
- Efield contribution within 10% of calculation based on equilibrium radial distribution
- Pitch contribution within 5% of calculation based on vertical distribution

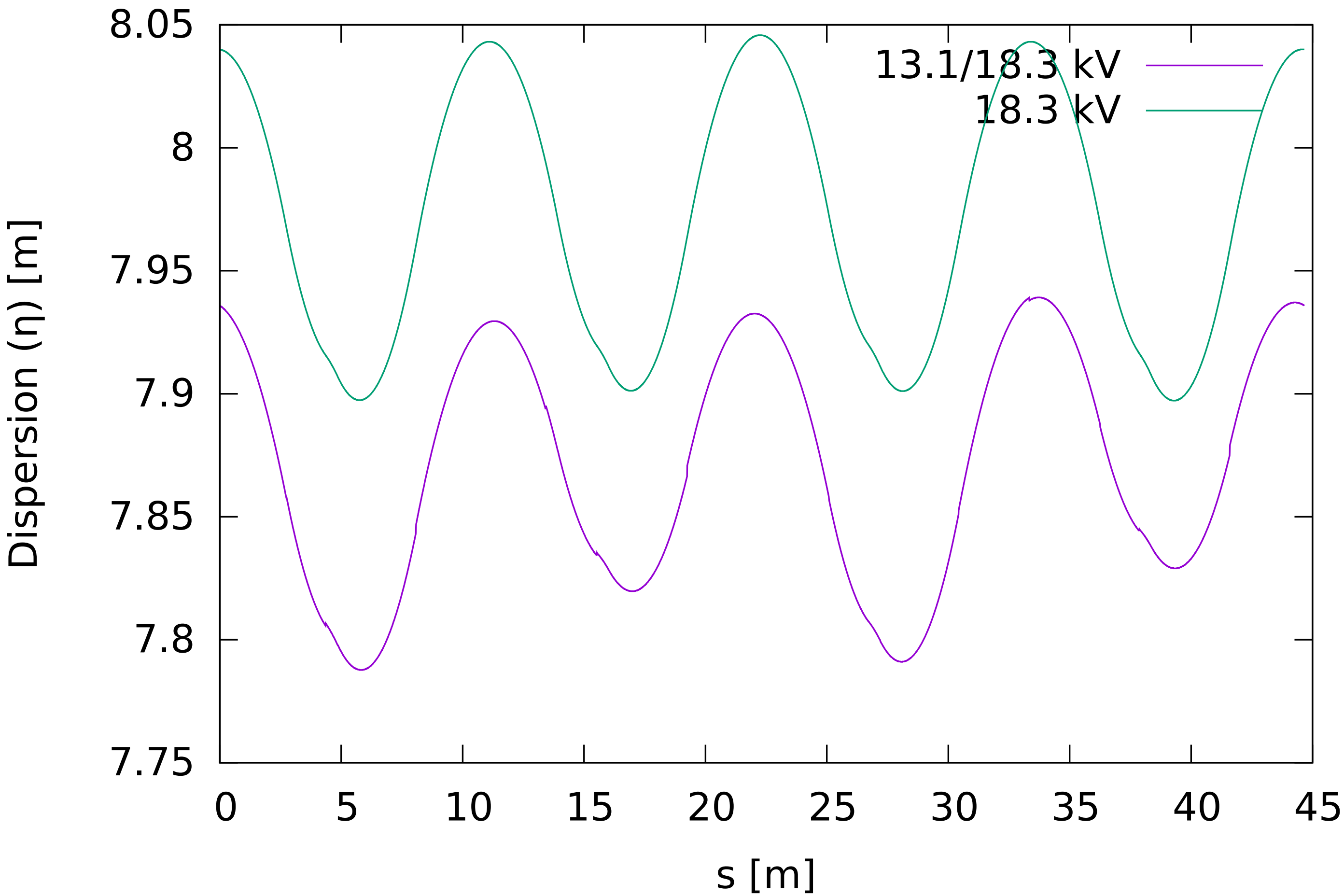
- Next step

  Extract equilibrium radial distribution with FR-analysis and compare with simulation

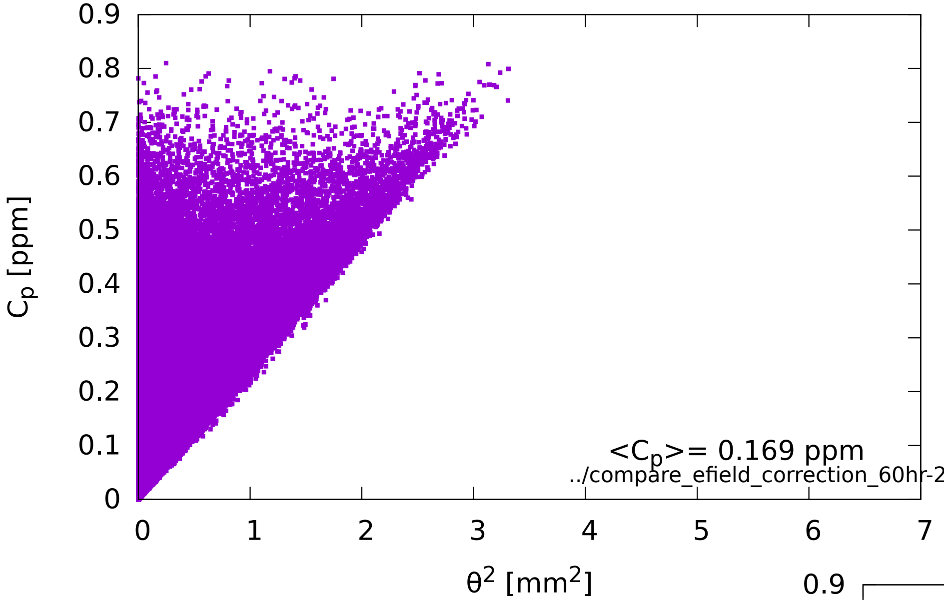
  And to understand the 10 and 5% discrepancies

END

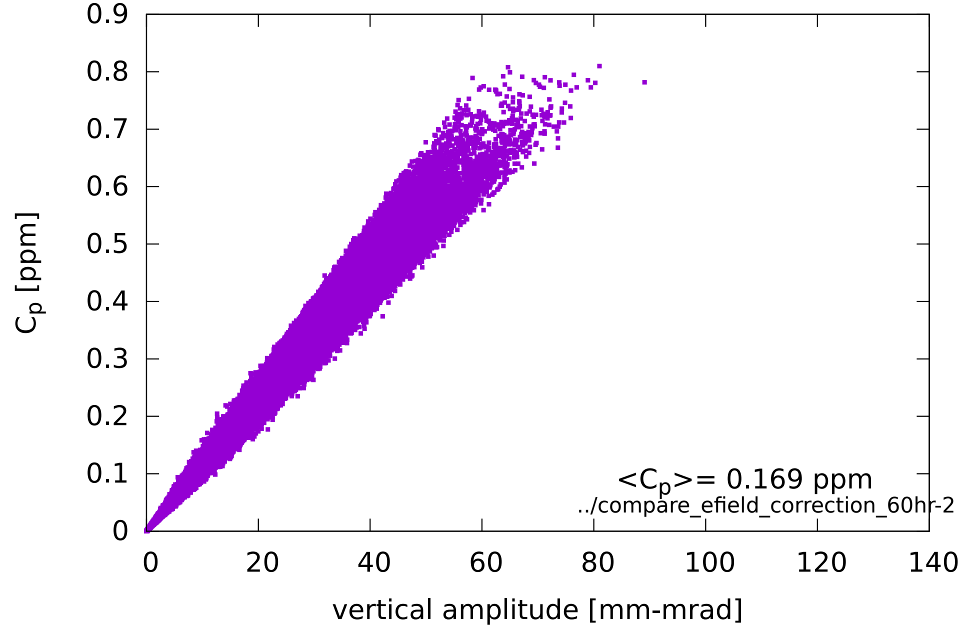

18.1 kV
Dispersion (η) [m]
s [m]
8.06
8.04
8.02
8
7.98
7.96
7.94
7.92
7.9
7.88
0
5
10
15
20
25
30
35
40
45



8.05
8
7.95
7.9
7.85
7.8
7.75
0
5
10
15
20
25
30
35
40
45
Dispersion (η) [m]
s [m]
13.1/18.3 kV
18.3 kV



0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
C_p [ppm]
θ^2 [mm^2]
<C_p>= 0.169 ppm
../compare_efield_correction_60hr-2



0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
20
40
60
80
100
120
140
C_p [ppm]
vertical amplitude [mm-mrad]
<C_p>= 0.169 ppm
../compare_efield_correction_60hr-2

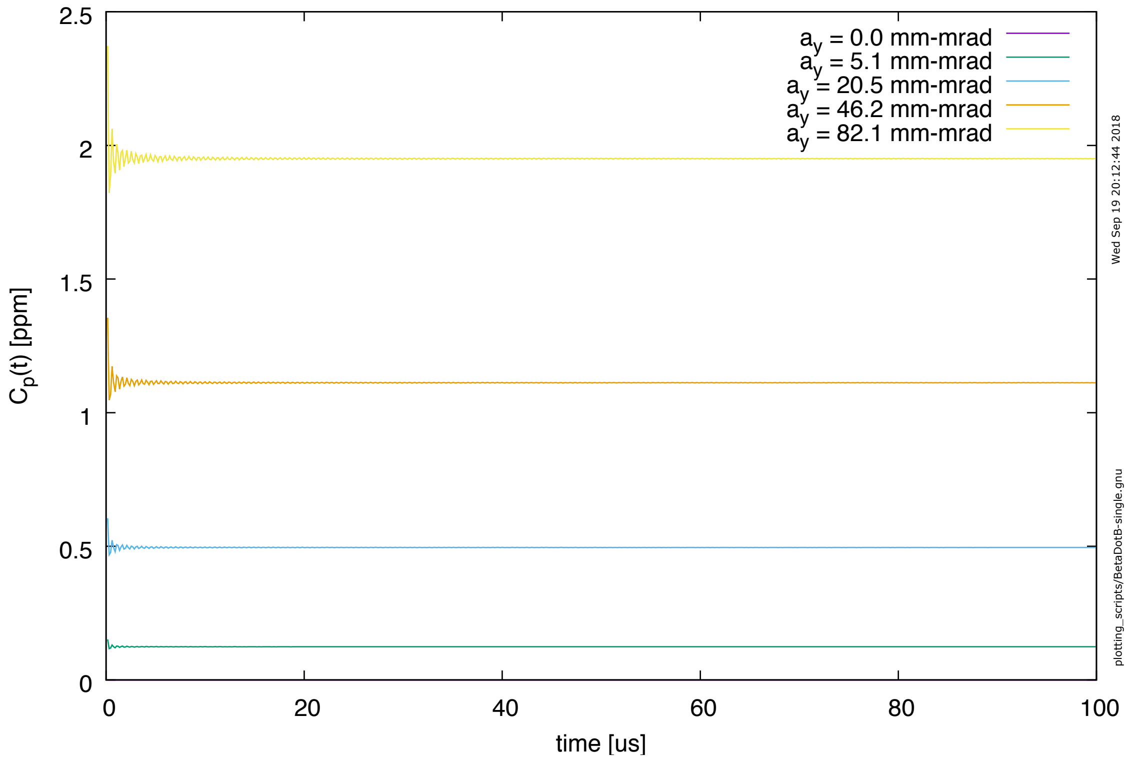
1. Generated and track a distribution and compute
    - Momentum and frequency distribution
    - E-field correction for each particle
    - Vertical phase space distribution and pitch correction
    - Fast rotation signal
2. Perform fast rotation analysis and see if we get the right answer.

Backup

How does pitch correction depend on betatron amplitude?

The initial displacements and amplitudes are

1. $y = 0$, $a_y = 0.0$ mm-mrad
2. $y = 1$ cm, $a_y = 5.12$ mm-mrad
3. $y = 2$ cm, $a_y = 64.3$ mm-mrad
4. $y = 3$ cm, $a_y = 46.2$ mm-mrad
5. $y = 4$ cm, $a_y = 82.1$ mm-mrad



Cartesian coordinates − no z- dependence

$$E_x - iE_y = (b_n - ia_n)\frac{(x+iy)^n}{r_0^n}$$

Satisfies Maxwell for any n

Not so in cylindrical coordinates. But we can expand as

$$E_x = \sum_n b_n \left(\frac{x}{r_0}\right)^n$$

In the midplane (y=0)

This was our guess

$$E \sim k\left((x - \frac{x^2}{2\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}}\right)$$

General form

$$E_x = \sum_n \frac{b_n}{r_0^n} x^n$$

$$\Rightarrow \frac{b_2/r_0^2}{b_1/r_0} = -\frac{1}{2\rho_0} = -0.0703\mathrm{m}^{-1}$$

The fitted values give

$$\Rightarrow \frac{b_2/r_0^2}{b_1/r_0} = -0.0593\mathrm{m}^{-1}$$

$$\nabla V = \mathbf{E} \sim k \left((x - \frac{x^2}{\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}} \right)$$

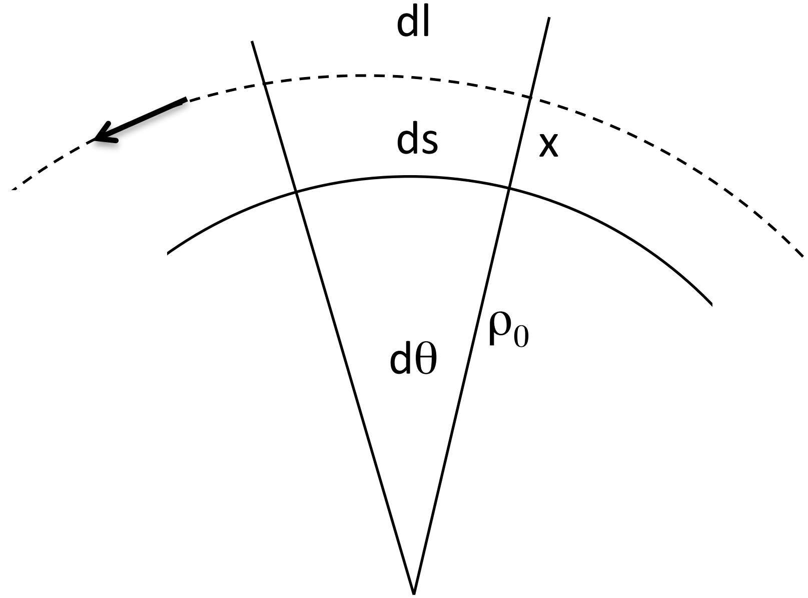
Linear term

Sextupole

The solution is not unique.
It is possible to find a solution that is linear in x,
but then it is necessarily nonlinear in z (vertical)

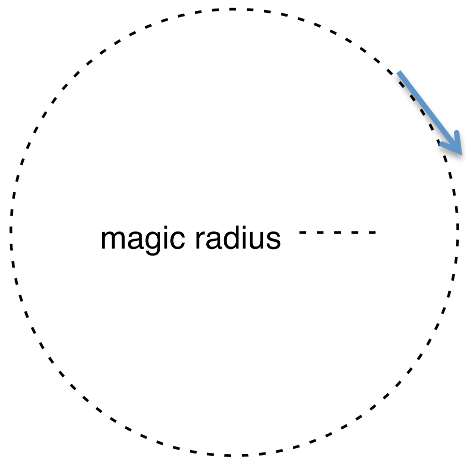
There is inevitably a sextupole component with curved plates independent of the plate shape details and alignment.

Path length

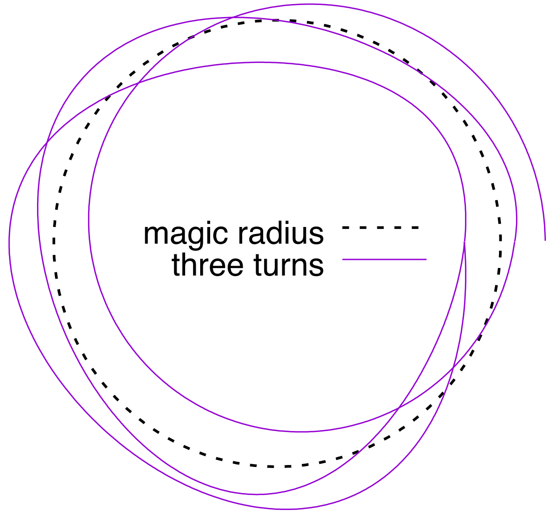


A particle oscillating about the magic radius (ρ_0) spends more time at x>0 than x<0

$$dl = (\rho_0 + x)d\theta$$



The E-field along the trajectory at the magic radius (momentum = p_0) is zero.



But what about the muon with momentum p_0 that oscillates about the magic radius with some betatron amplitude x_β ?

Or the muon with momentum $p_0 + \Delta p$ and betatron amplitude x_β?

$$x = \eta\delta + x_\beta \qquad \delta = \Delta p/p_0$$

How does E-field correction depend on betatron amplitude?
Consider 4 distinct trajectories,
initialized at t=0 with

$$x' = 0, \Delta p/p = 0.2\%$$

The initial displacements and betatron amplitudes are

1. x = 0.02 m, a_x = 0.738 mm-mrad
2. x = 0.0 m, a_x = 39.7 mm-mrad
3. x = -0.01 m, a_x = 64.3 mm-mrad
4. x= -0.02 m, a_x = 97.66 mm-mrad

Note that the betatron coordinates are related to x and x' according to

$$\begin{aligned} x_\beta &= x - \eta(\Delta p/p) \\ x'_\beta &= x' - \eta'(\Delta p/p) \end{aligned}$$

And $a_x = (\gamma x_\beta^2 + 2\alpha x_\beta x'_\beta + \beta {x'_\beta}^2)^{1/2}$ is the betatron amplitude

E-field correction

$$C_e = \left(1 - \frac{1}{a_\mu}\frac{m^2}{p^2}\right)\frac{\beta E_r}{cB} \sim \left(1 - \frac{m^2}{a_\mu p_0^2}(1 - 2\frac{\Delta p}{p})\right)\frac{\beta E_r}{cB} \qquad (1)$$

Magic momentum $m^2/p_0^2 = a_\mu$

$$x_e = \eta\delta$$

$$C_e(\delta, x_{\beta 0}) \quad \approx \quad 2\frac{\beta k}{cB}\left(\frac{x_e^2}{\eta} + \frac{1}{2\rho_0}(\frac{x_e^3}{\eta} + \frac{1}{2}x_{\beta 0}^2\frac{x_e}{\eta})\right)$$

If $\langle x_e \rangle = \langle \delta \rangle \eta = 0$ then correction is independent of x_β

$$\langle C_e(\delta, x_{\beta 0})\rangle \quad = \quad 2\frac{\beta k}{cB}\eta\langle\delta^2\rangle$$

$$\langle C_e(\delta, x_{\beta 0})\rangle = 2\frac{\beta k}{cB}\eta\langle\delta^2\rangle$$

$$k = (22.409\frac{V}{27.2} \times 10^6\mathrm{V/m}^2$$

$$\eta = 8.3\mathrm{m}$$

$$2\frac{(0.999)(22.409 \times 10^6)\frac{V}{V_0}}{(3 \times 10^8)(1.45)}8.3(0.001)^2 = 85.429 \times 10^{-8}$$

$$k_{eff} = k\frac{L_{quad}}{2\pi R_0} = k\frac{156}{360} = (0.43333)k$$

$$\eta_{eff} = \langle\eta\rangle$$

$$k(\mathrm{MV/m}^2) = \frac{22.409}{27.2}V(\mathrm{kV})$$

If $\langle x_e \rangle = \langle \delta \rangle \eta \neq 0$ then according to the Miller/Nguyen rule

To minimize the E-field correction choose p_0 so that

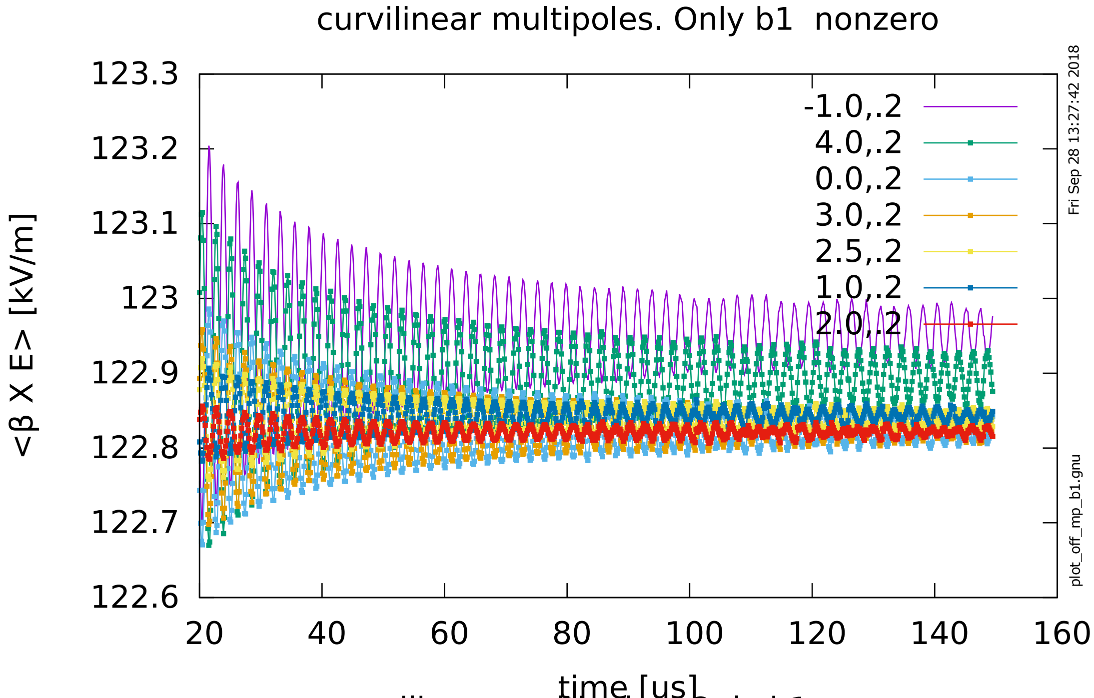
$$2a_\mu \langle \frac{p - p_0}{p_0} \rangle = \frac{m^2}{p_0^2} - a_\mu$$

Then

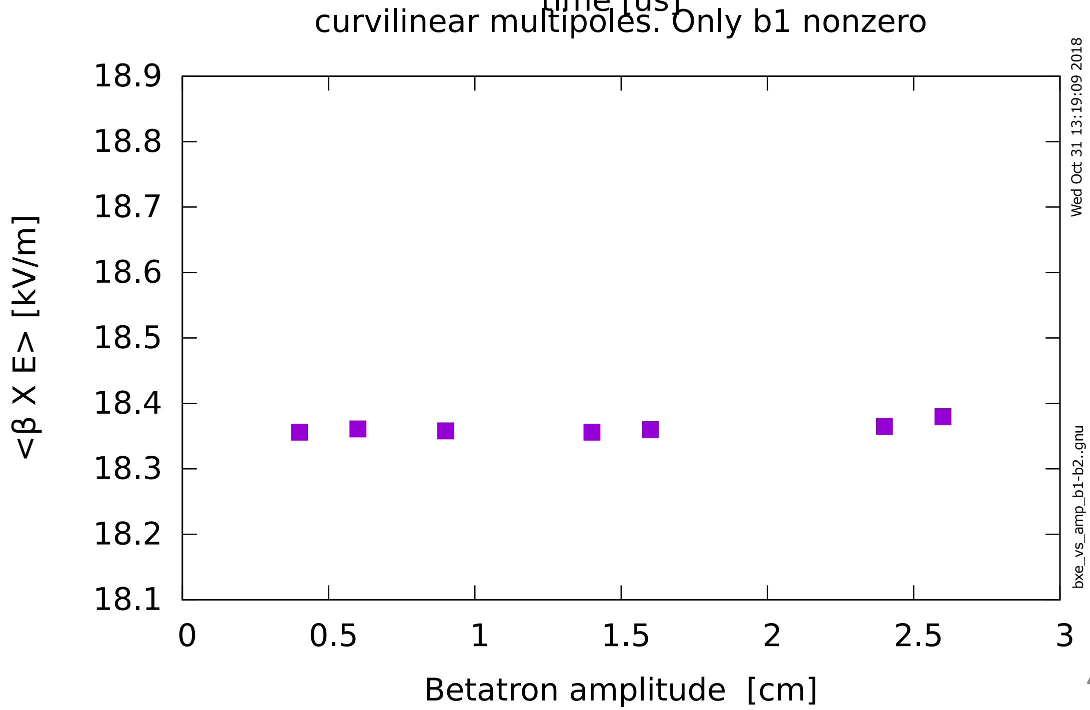
$$\langle C_e \rangle \sim 2 \left[-\eta(\langle \delta \rangle^2 - \langle \delta^2 \rangle) + \frac{1}{4\rho_0}(\langle \delta \rangle^2 \langle x_{\beta 0}^2 \rangle \right] \frac{\beta k}{cB}$$

Contribution from betatron amplitude

Correction *increases* with betatron amplitude

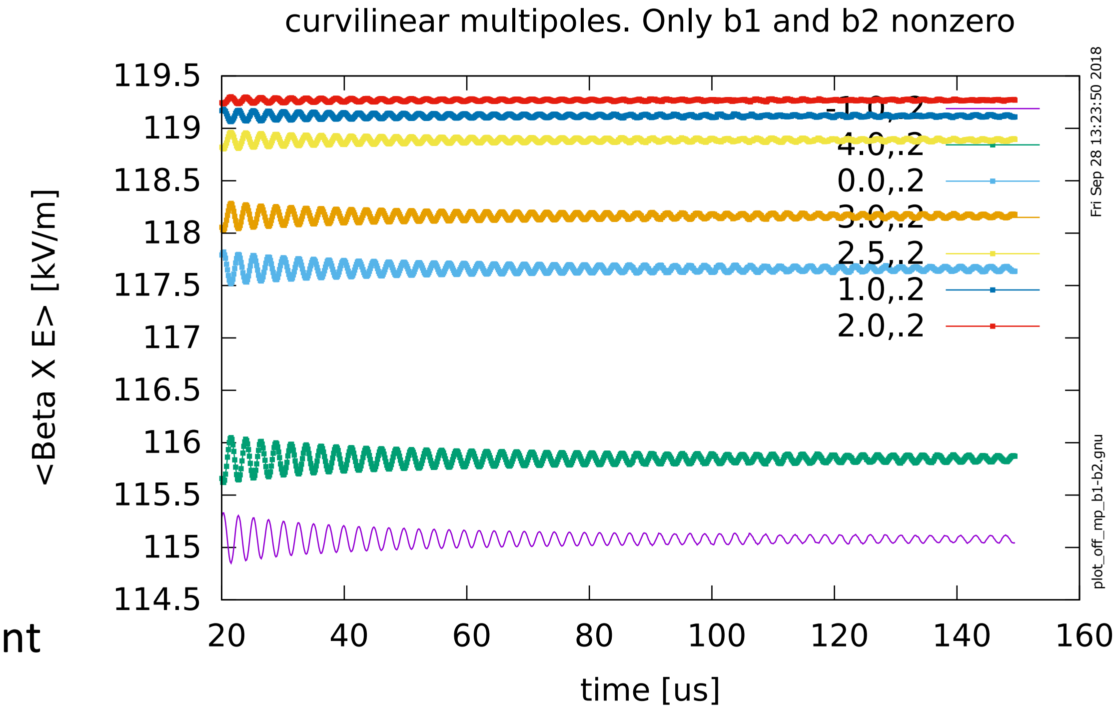


Turn off sextupole component



Correction increases with amplitude (path length effect)

Restore sextupole component



Correction *decreases* with amplitude (average orbit shifted radially inward)