Dependence of Momentum Acceptance on Displacement of Injected Muons, and the Crnkovic Differential Decay Effect

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Correlation between muon spin and momentum in muon rest frame

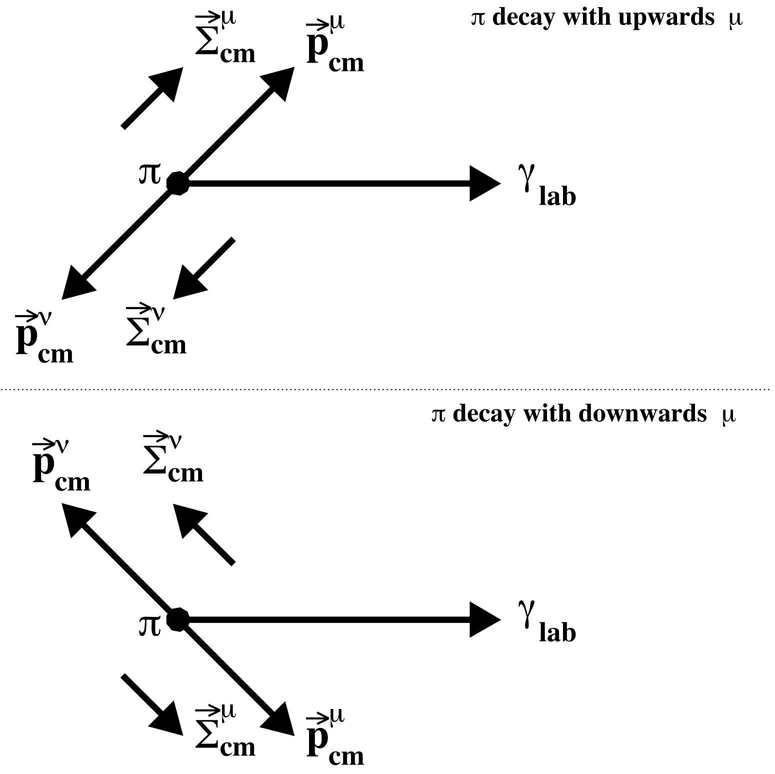


Figure 1: Cartoon illustrating the correlation between muon momentum and spin in the pion rest frame.

Spin angle lags for muon x’ > 0

Spin angle leads for muon x’ < 0

Doc-db 3841 Crnkovic, et al.
Doc-db 27609 Morse

Crnkovic et al. doc db 3477 and 3841 show that spin angle and muon phase space coordinates x, x’, are correlated in muons propagated to end of M5

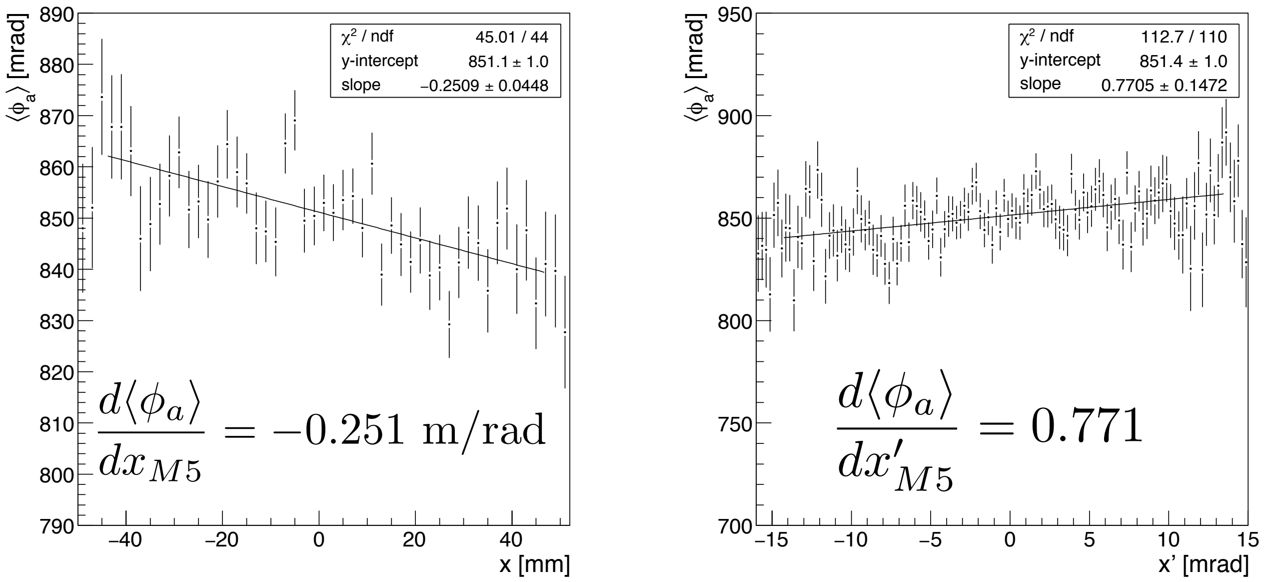
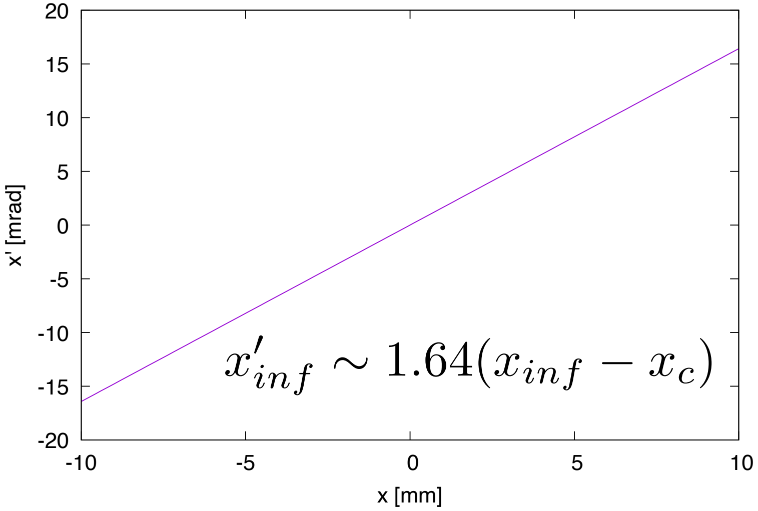


FIG. 1: MC simulation of average z-x spin angle at end of M5[2]

=>

Extrapolation to inflector exit



$x_c = 77 \text{ mm}$

If x, x’ are correlated with momentum acceptance, we expect early to late effect

$$\langle\Delta\omega_a\rangle = \frac{d\langle\phi_a\rangle}{d\gamma}\frac{d\langle\gamma\rangle}{dt} = \frac{d\langle\phi_a\rangle}{dx_{inf}}\frac{dx_{inf}}{d\gamma}\frac{d\langle\gamma\rangle}{dt} = \frac{\langle\phi_a\rangle}{dx_{inf}}\frac{dx_{inf}}{d\delta}\frac{d\delta}{d\gamma}\frac{d\langle\gamma\rangle}{dt} = \frac{\langle\phi_a\rangle}{dx_{inf}}\frac{dx_{inf}}{d\delta}\frac{1}{\gamma}\frac{d\langle\gamma\rangle}{dt}$$

doc db 3841

Injection Acceptance Model

Momentum acceptance in terms of initial conditions $x_{inf}, x'_{inf}, n,$ and the kick θ_k

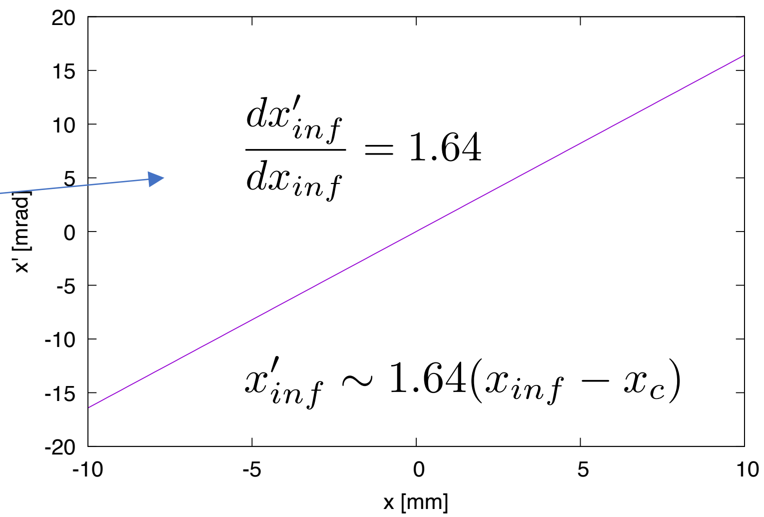
$$\rightarrow -\frac{1}{2}\left(A-(x_{inf}-\overline{\theta_k}\beta)-\frac{(\overline{x'}_{inf}\beta)^2}{A+(x_{inf}-\overline{\theta_k}\beta)}\right)<\eta\delta<\frac{1}{2}\left(A+(x_{inf}-\overline{\theta_k}\beta)-\frac{(\overline{x'}_{inf}\beta)^2}{A-(x_{inf}-\overline{\theta_k}\beta)}\right)$$

$$\left\{\begin{array}{l}\overline{x'_{inf}} \equiv x'_{inf}+\theta_k\cos(Q_x\frac{\pi}{2}) \\ \overline{\theta_k}\equiv\theta_k\sin(Q_x\frac{\pi}{2})\end{array}\right.$$

A is the collimator aperture

doc-db 27471-v1 & 26243-v2

Phase space at inflector exit



Given an injected distribution $\rho_{ini}(x, x', \delta)$ we can generate captured distribution $\rho_{cap}(x, x', \delta)$

For our initial distribution assume

- x_{inf} gaussian distributed over the range $68 \text{ mm} < x_{inf} < 86 \text{ mm}$
- $x'_{inf} \sim 1.64(x_{inf} - x_c)$
- Momentum (δ), independent of x_{inf}

$$\rightarrow -\frac{1}{2}\left(A - (x_{inf} - \overline{\theta_k}\beta) - \frac{(\overline{x'}_{inf}\beta)^2}{A + (x_{inf} - \overline{\theta_k}\beta)}\right) < \eta\delta < \frac{1}{2}\left(A + (x_{inf} - \overline{\theta_k}\beta) - \frac{(\overline{x'}_{inf}\beta)^2}{A - (x_{inf} - \overline{\theta_k}\beta)}\right)$$

Assume uniform distribution of momenta in injected beam

Scan x_{inf} over inflector width (from 0.068m to 0.086m)

Use $x'_{inf} \sim 1.64(x_{inf} - x_c)$

Use measured kick pulse shape to get θ_k

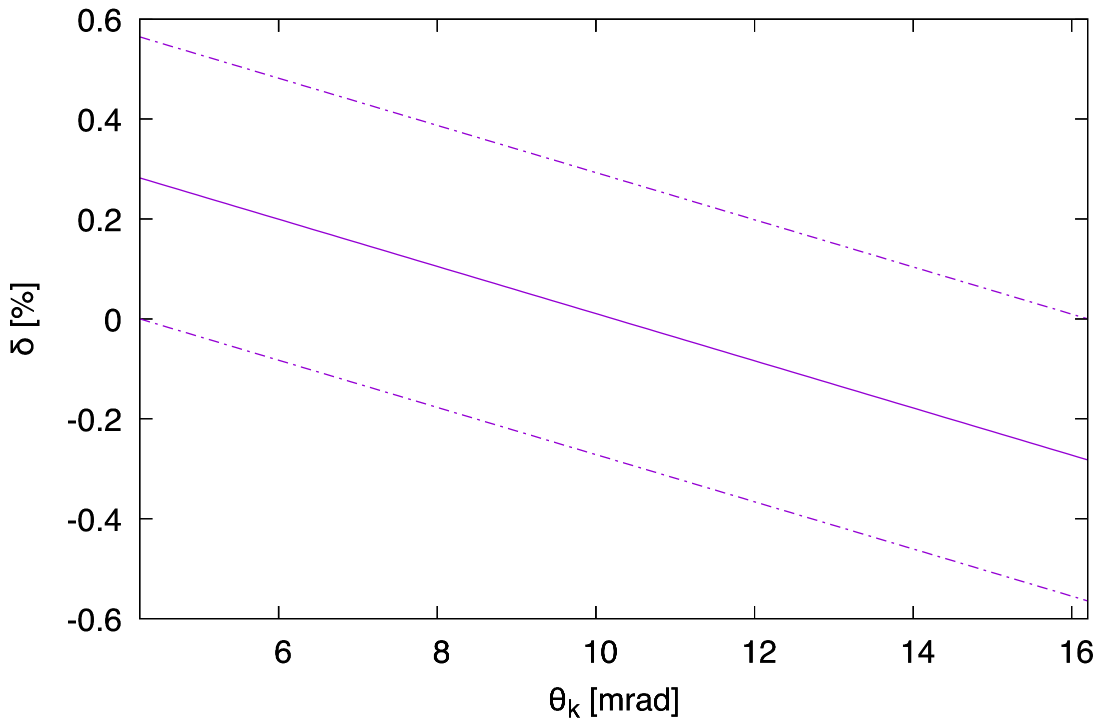
Weight by measured pulse intensity and assume x is gaussian distributed

Simplest Case

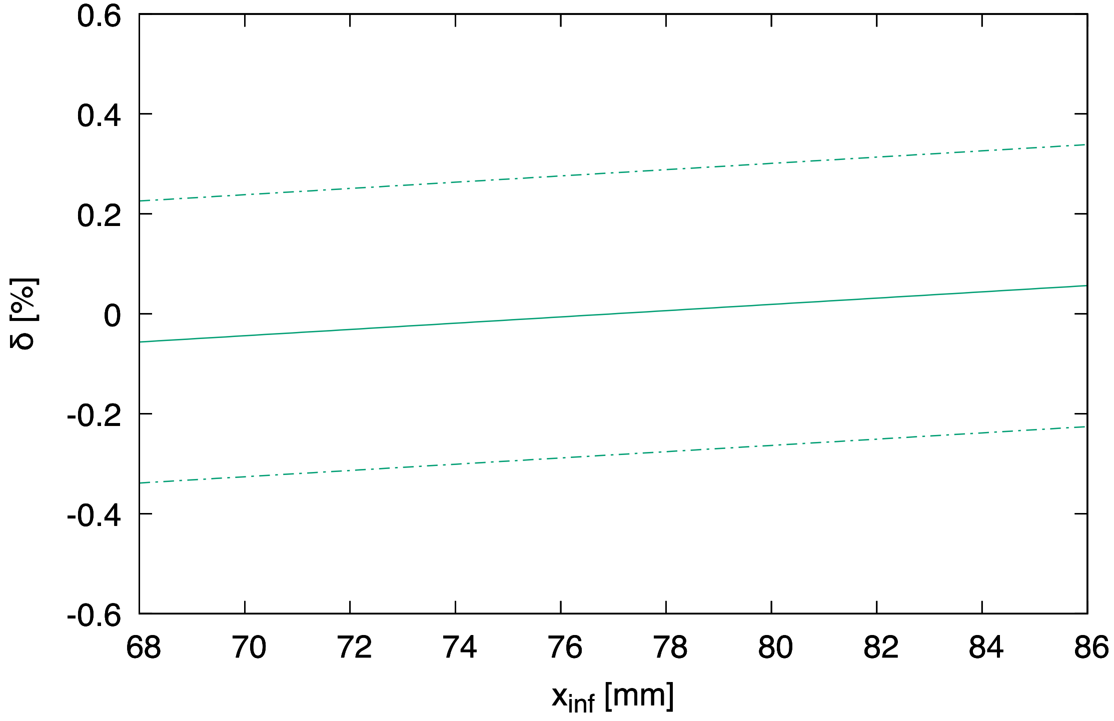
$$\rightarrow -\frac{1}{2}\left(A-(x_{inf}-\overline{\theta_k}\beta)-\frac{(\overline{x'}_{inf}\beta)^2}{A+(x_{inf}-\overline{\theta_k}\beta)}\right)<\eta\delta<\frac{1}{2}\left(A+(x_{inf}-\overline{\theta_k}\beta)-\frac{(\overline{x'}_{inf}\beta)^2}{A-(x_{inf}-\overline{\theta_k}\beta)}\right)$$

In the limit where $x'_{inf}=0$, $\sigma_x = \infty$

Momentum acceptance as a function of kick angle for $x_{inf} = 77$ mm, $x'_{inf} = 0$

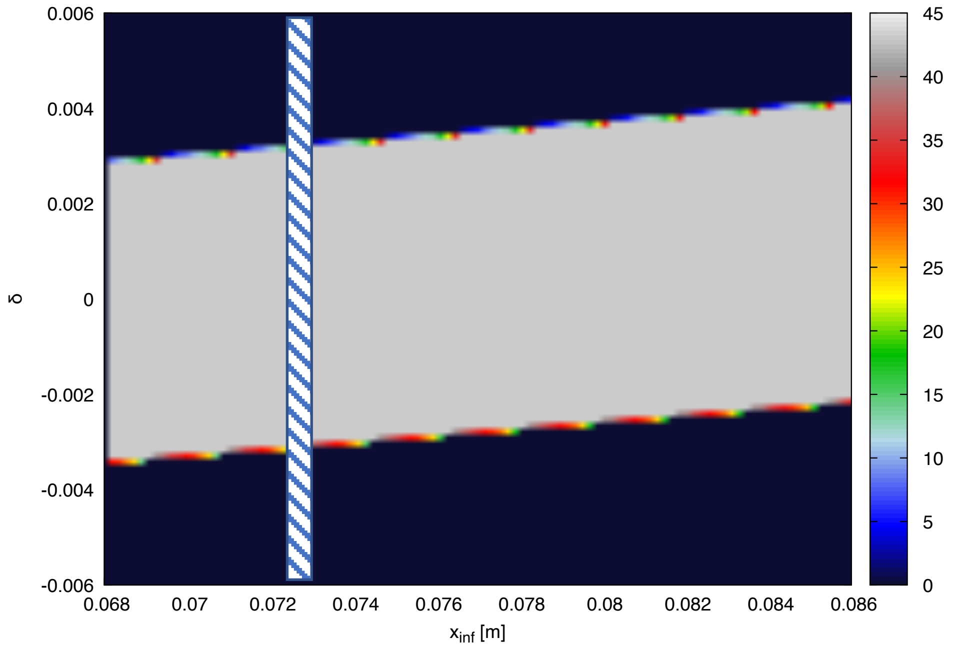


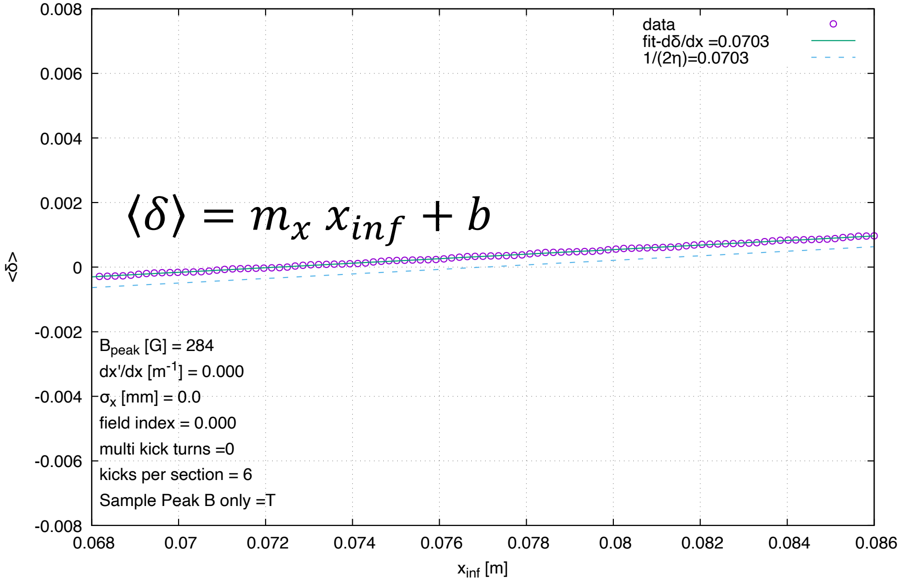
Momentum acceptance as a function of x_{inf} for $\theta_k = 10.2$ mrad, $x'_{inf} = 0$



If

- $\sigma_x = \infty$
- $dx'/dx = 0 \quad (x'_{inf}=0)$
- Field Index =0
- Single kick
- B_{peak} =284 G





$$\langle \delta(x) \rangle = \int \delta(x)\rho(x,\delta)d\delta$$

$$\chi^2 = \sum_{ij} \left(\delta_j - (m_x \cdot x_i + b)\right)^2 W_{ij}$$

$$\frac{d\langle\delta\rangle}{dx_{inf}} = m_x$$

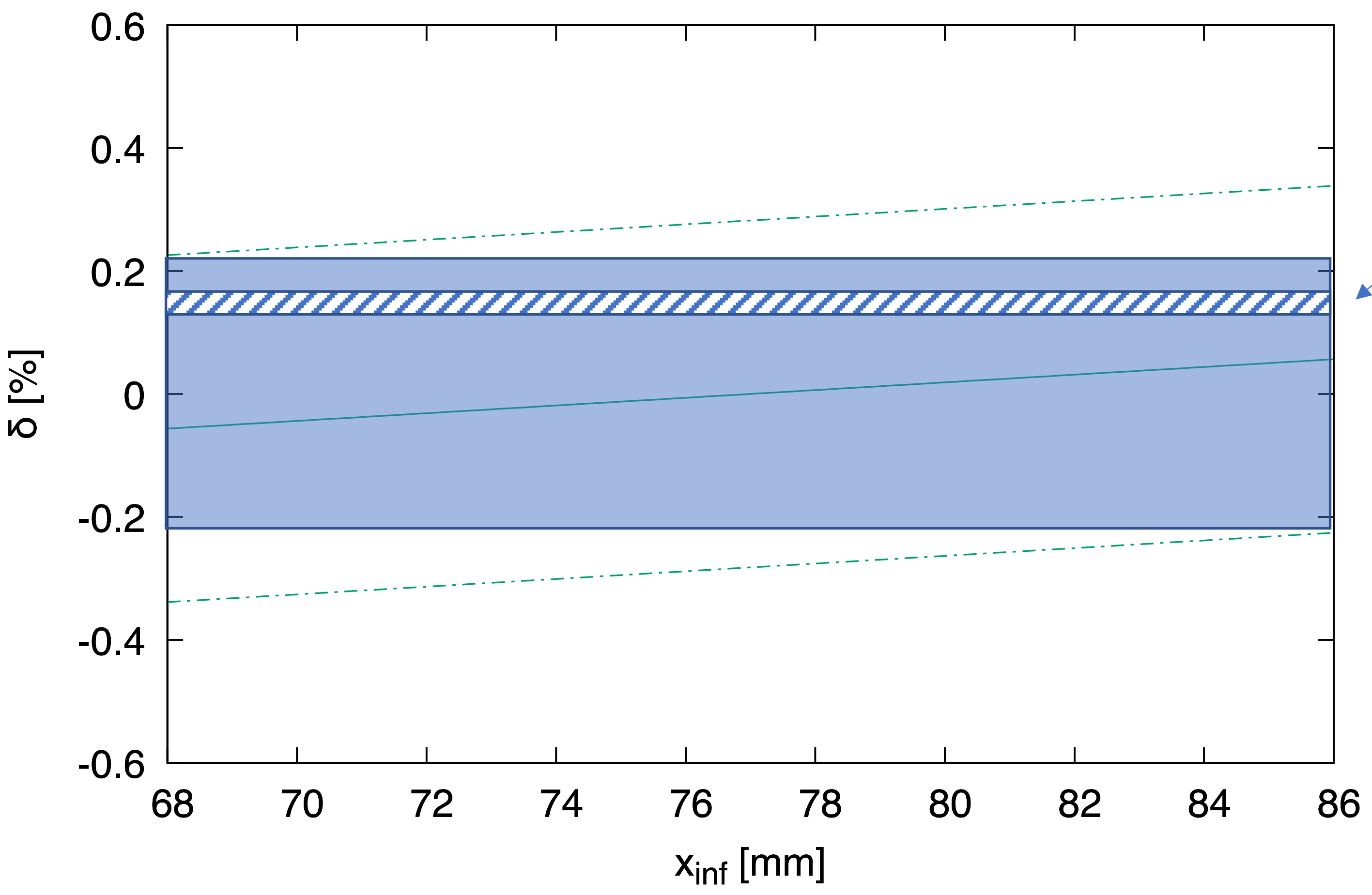
We have $$\frac{d\langle\delta\rangle}{dx_{inf}} = m_x$$

But what we want is

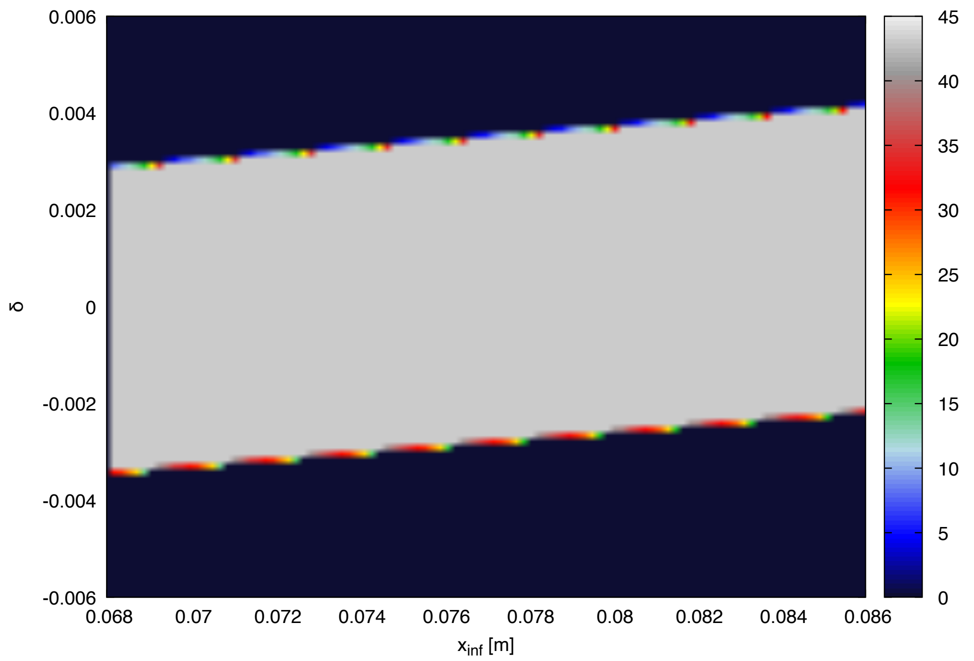
$$\langle\Delta\omega_a\rangle = \frac{d\langle\phi_a\rangle}{dx_{inf}}\frac{d\langle x_{inf}\rangle}{d\delta}\frac{1}{\gamma}\frac{d\langle\gamma\rangle}{dt}$$

$$\frac{d\langle x_{inf}\rangle}{d\delta}$$

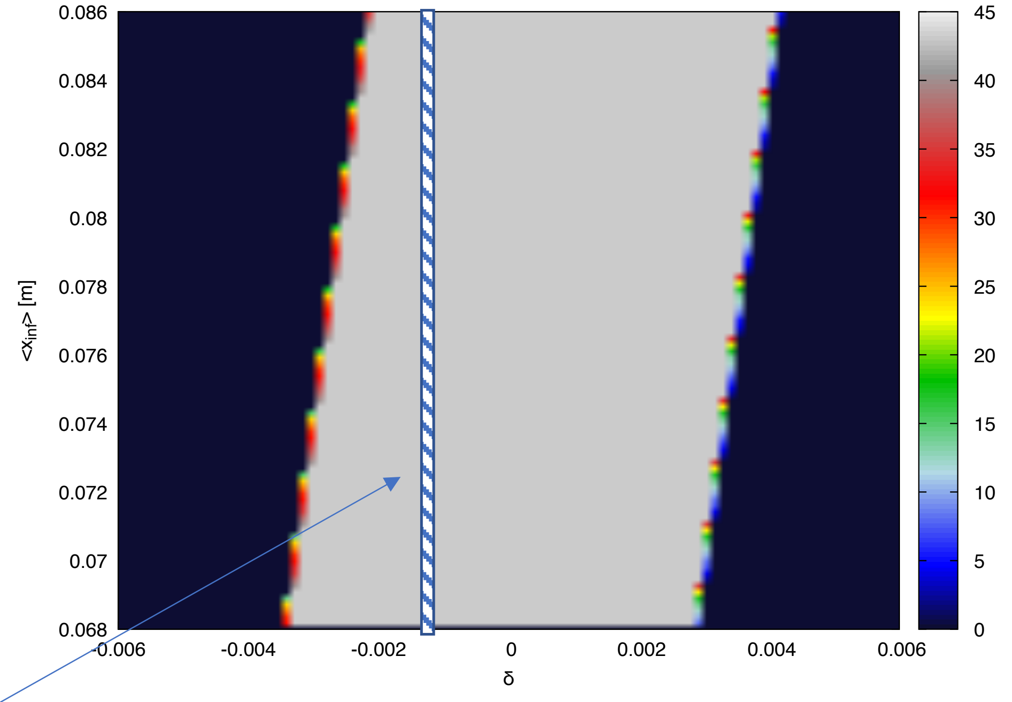
Momentum acceptance as a function of x_{inf} for θ_k = 10.2 mrad, x'_{inf} =0



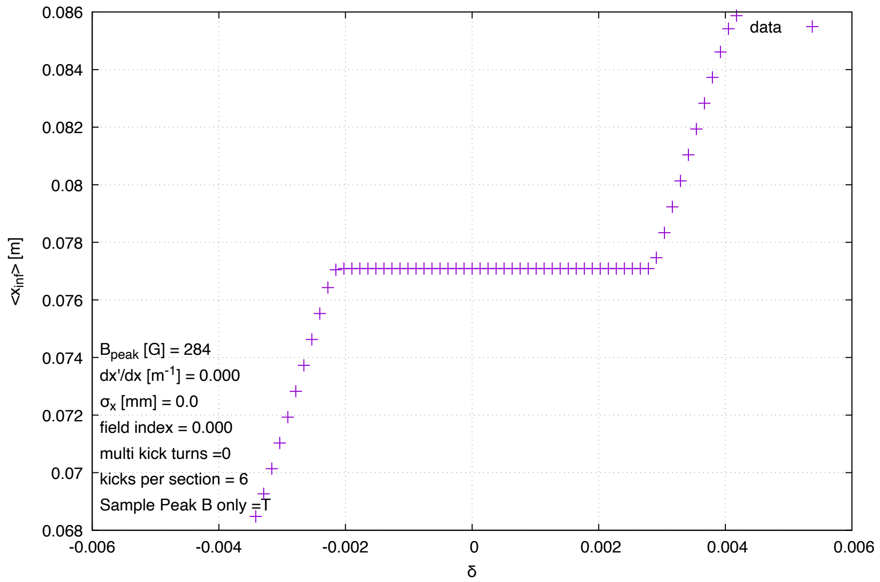
The momentum δ in the slice is independent of x_{inf}



Same data
Axis swapped



$$\langle x_{inf}(\delta)\rangle = \int x(\delta)\rho(x,\delta)dx$$



Note that

$\langle x_{inf}(\delta)\rangle$ *is independent of* δ, for $-0.002 \lesssim \delta \lesssim 0.002$

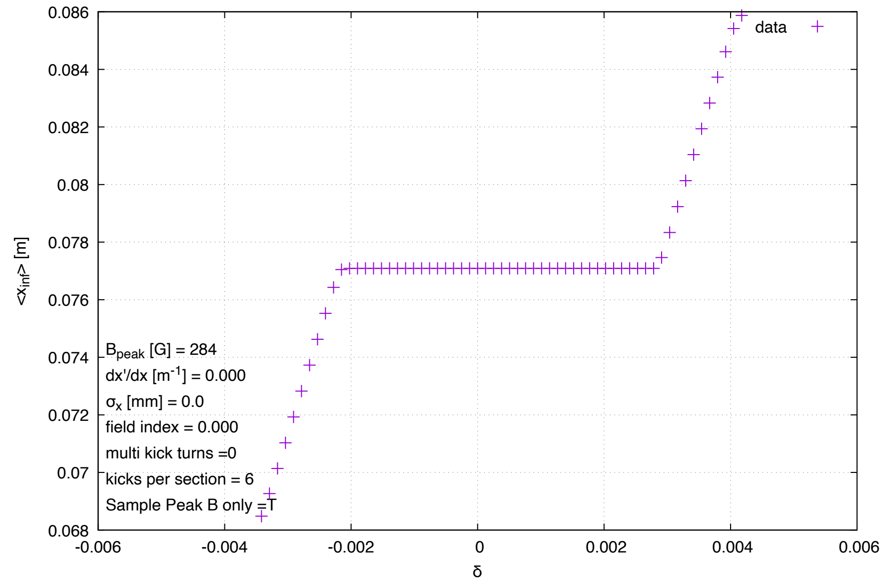
Evidently

$$1/\left(\frac{d\langle\delta\rangle}{dx_{inf}}\right) \neq \frac{d\langle x_{inf}\rangle}{d\delta}$$

$\frac{d\langle x_{inf}\rangle}{d\delta}$ Is the appropriate measure of the dependence of x_{inf} on δ

Recalling that $x_{inf} \Leftrightarrow \langle\phi_a\rangle$ we find that

$\langle\phi_a(\delta)\rangle$ *is independent of* δ, for $-0.002 \lesssim \delta \lesssim 0.002$



If

- $\sigma_x = \infty$
- dx'/dx =0 (x'$_{inf}$=0)
- Index =0
- Single kick
- B_{peak} =284 G

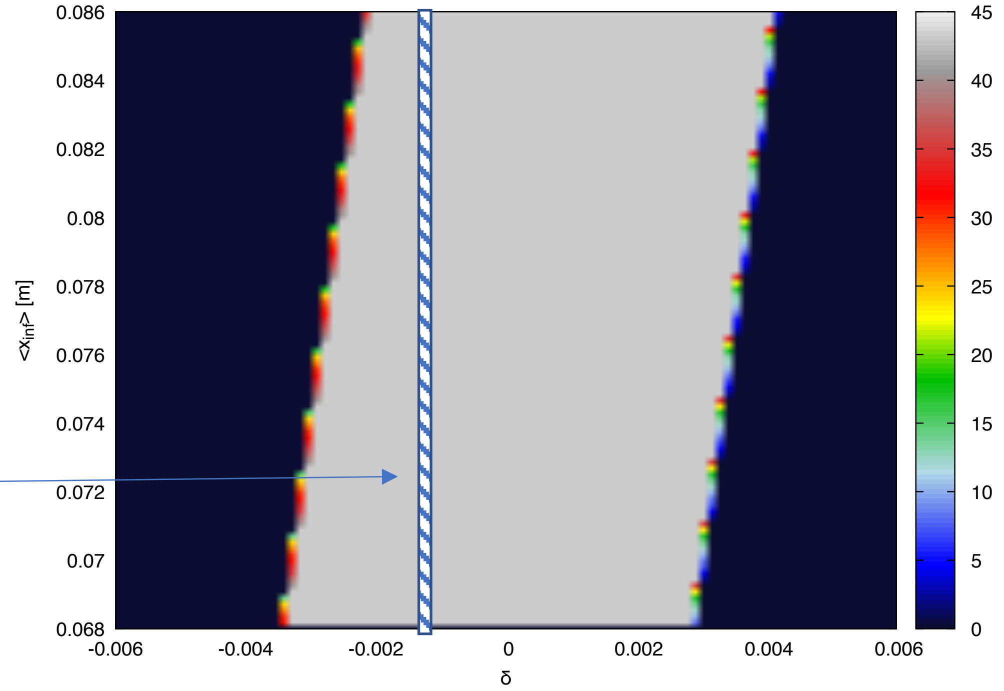
$$\langle x_{inf}(\delta)\rangle = \int x(\delta)\rho(x,\delta)dx$$

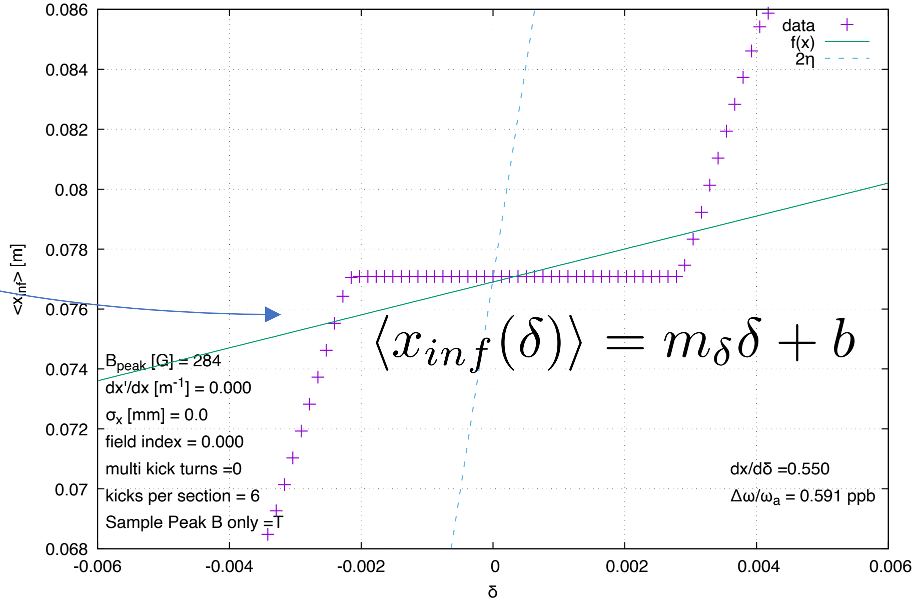
$$\chi^2 = \sum_{ij} \left(x_i - (m_\delta \cdot \delta_j + b)\right)^2 W_{ij}$$

$$\frac{d\langle x_{inf}\rangle}{d\delta} = m_\delta$$

$$\langle \omega_a \rangle = \frac{\langle \phi_a \rangle}{dx_{inf}} \frac{d\langle x_{inf}\rangle}{d\delta} \frac{1}{\gamma} \frac{d\langle \gamma \rangle}{dt}$$

$$\frac{\Delta\omega}{\omega} = 0.6 \text{ ppb}$$

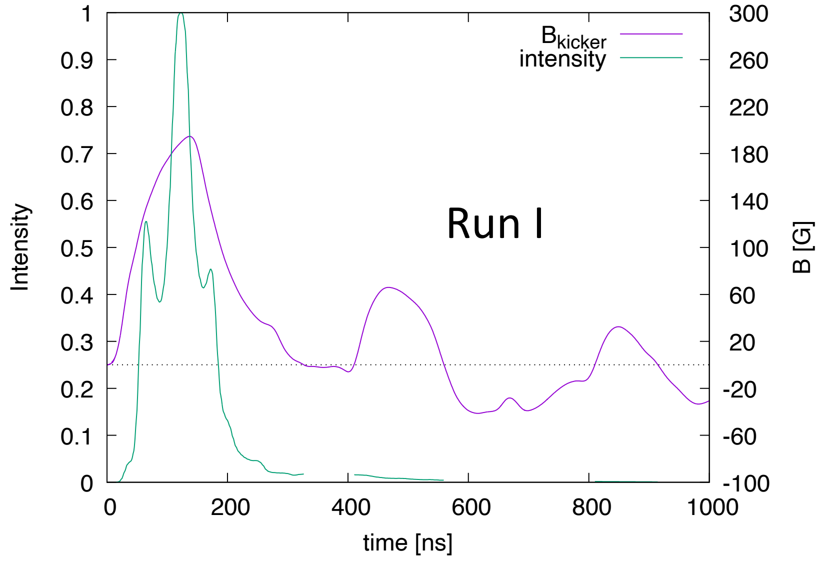




Re-cap

Assume uniform distribution of momenta in injected beam
Scan x_{inf} over inflector width (from 0.068m to 0.086m)
Use $x'_{inf} \sim 1.64(x_{inf} - x_c)$
Use measured kick pulse shape to get θ_k
Weight by measured pulse intensity and assume x is gaussian distributed

$$\rightarrow -\frac{1}{2}\left(A - (x_{inf} - \overline{\theta_k}\beta) - \frac{(\overline{x'}_{inf}\beta)^2}{A + (x_{inf} - \overline{\theta_k}\beta)}\right) < \eta\delta < \frac{1}{2}\left(A + (x_{inf} - \overline{\theta_k}\beta) - \frac{(\overline{x'}_{inf}\beta)^2}{A - (x_{inf} - \overline{\theta_k}\beta)}\right)$$



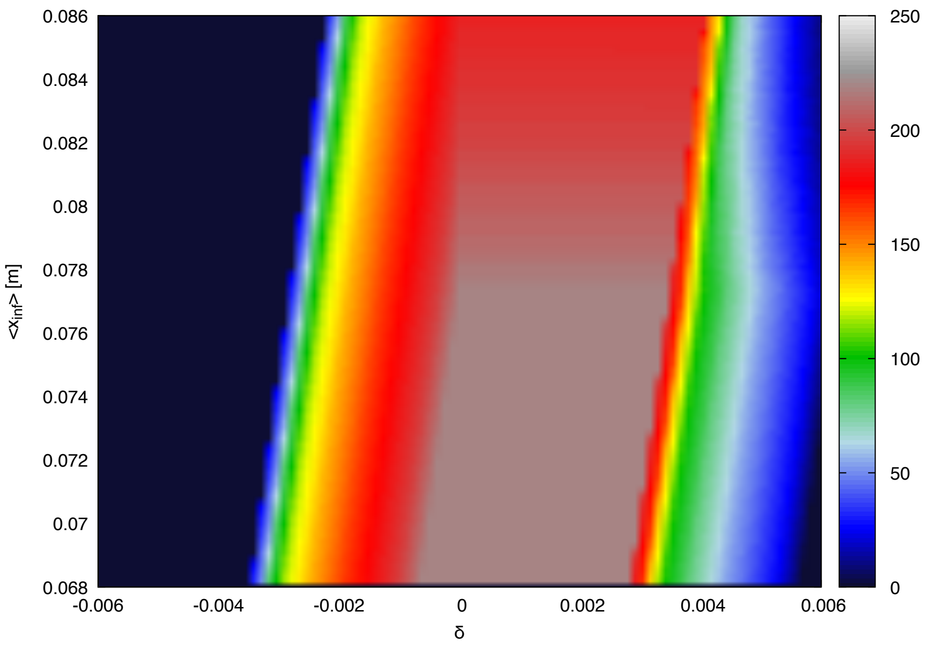
Time dependence of kicker field and injected pulse

$$\overline{x'_{inf}} = x'_{inf} + \sum_n \theta_k^n \cos(2\pi\nu[\frac{1}{4} + n])$$

$$\overline{\theta_k} = \sum_n \theta_k^n \sin(2\pi\nu[\frac{1}{4} + n])$$

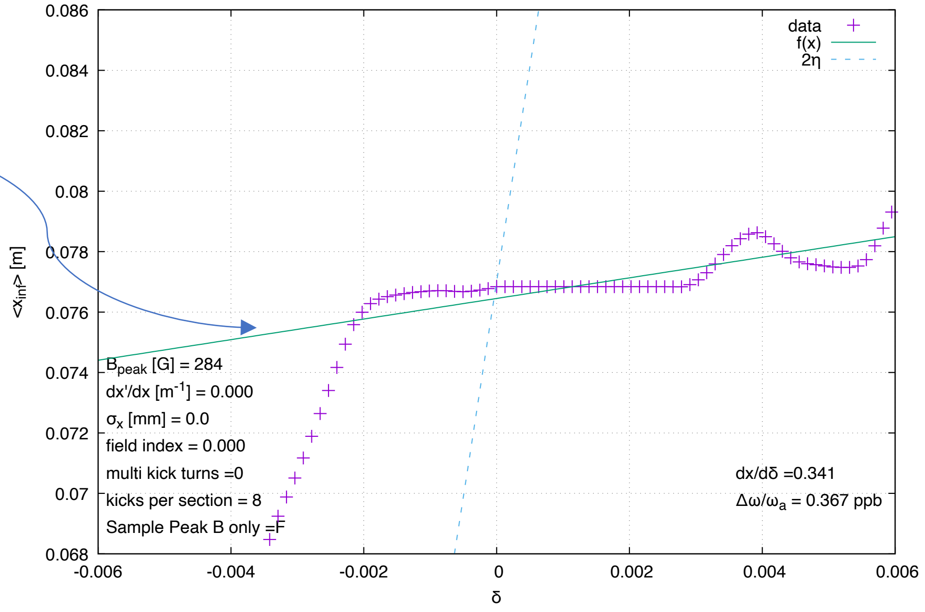
If

- $\sigma_x = \infty$
- dx'/dx =0 (x'_{inf}=0)
- Index =0
- Single kick
- B_{peak} =284 G – measured pulse shape



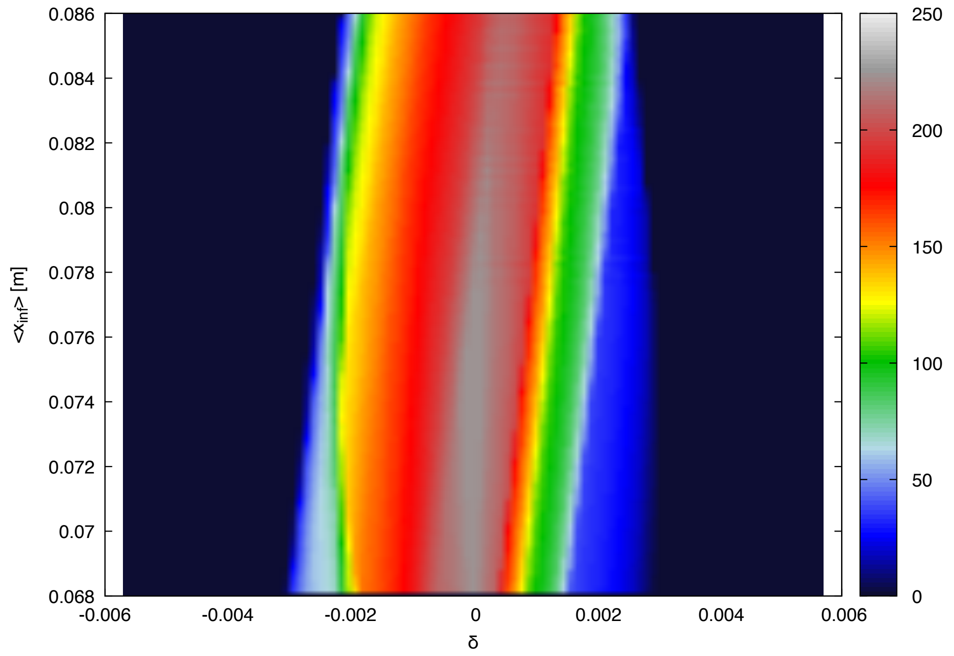
$$\frac{d\langle x_{inf}\rangle}{d\delta} = m_\delta$$

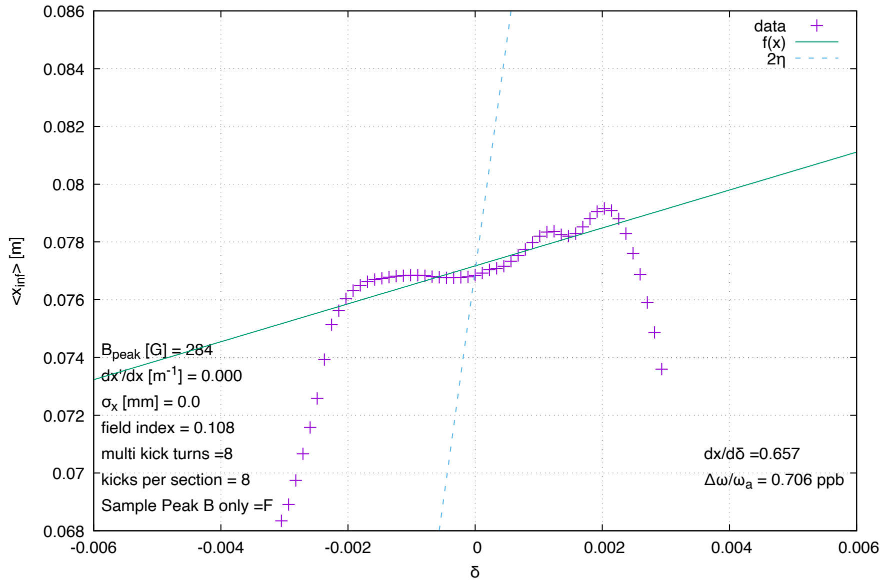
$$\frac{\Delta\omega}{\omega} = 0.4 \text{ ppb}$$



If

- $\sigma_x = \infty$
- dx’/dx =0 (x’$_{inf}$=0)
- Index =0.108
- Kick on multiple turns
- B_{peak} =284 G – measured pulse shape



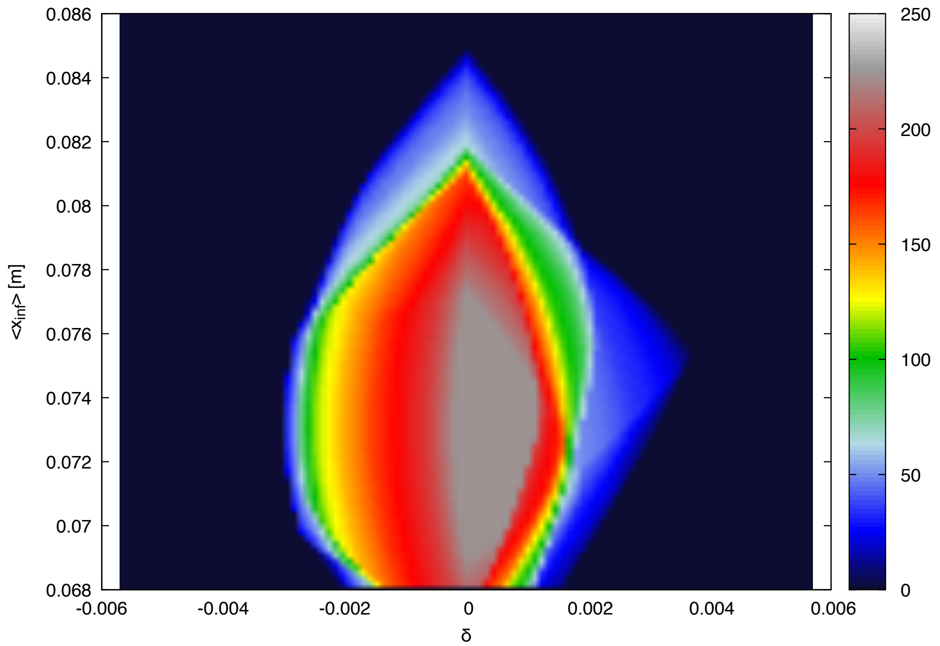


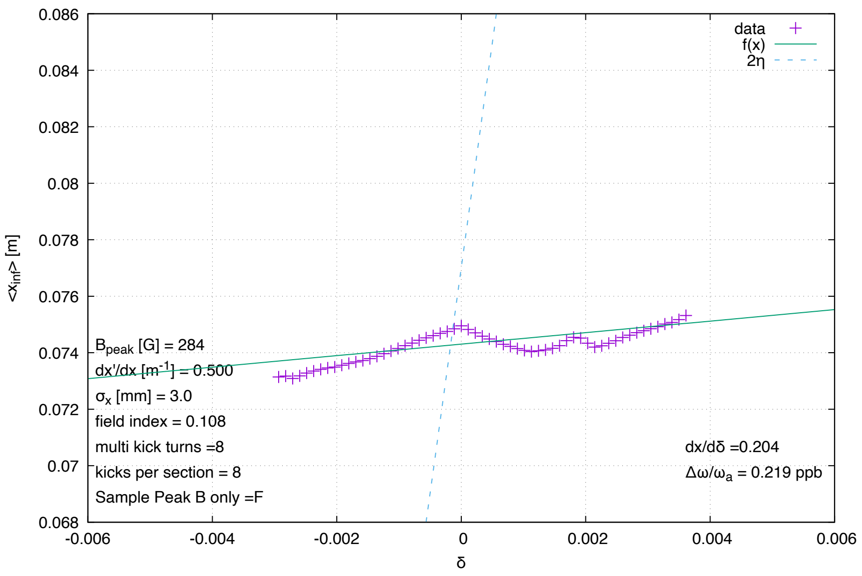
$$\frac{\Delta\omega}{\omega} = 0.7 \text{ ppb}$$

If

- $\sigma_x = 3$ mm
- dx'/dx = 0.5
- Index =0.108
- Kick on multiple turns
- B_{peak} =284 G – measured pulse shape

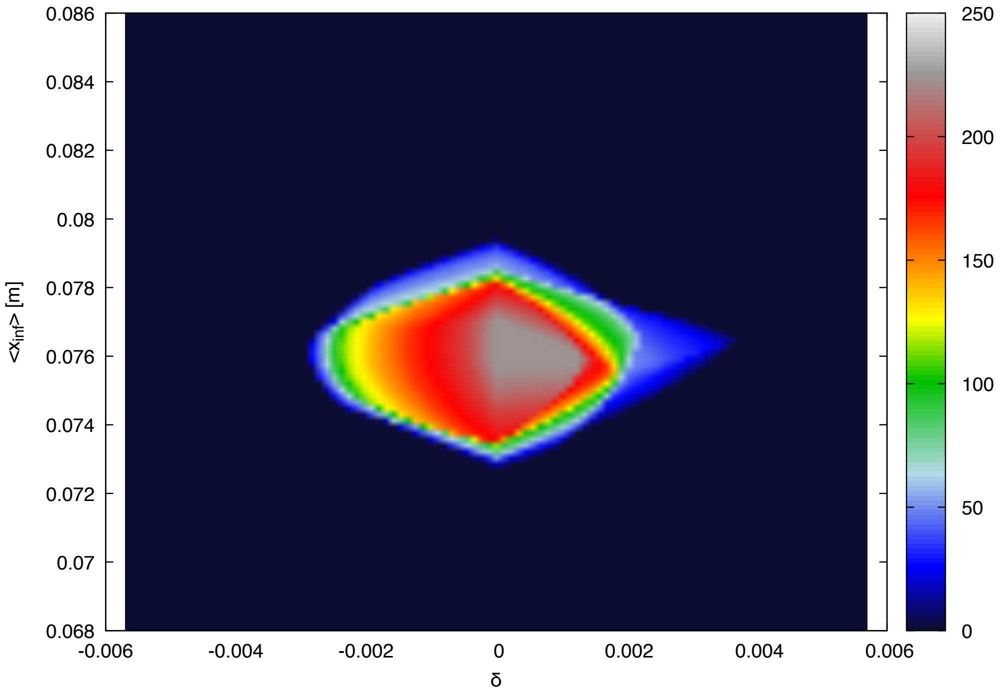
$$\frac{\Delta\omega}{\omega} = 0.2 \text{ ppb}$$



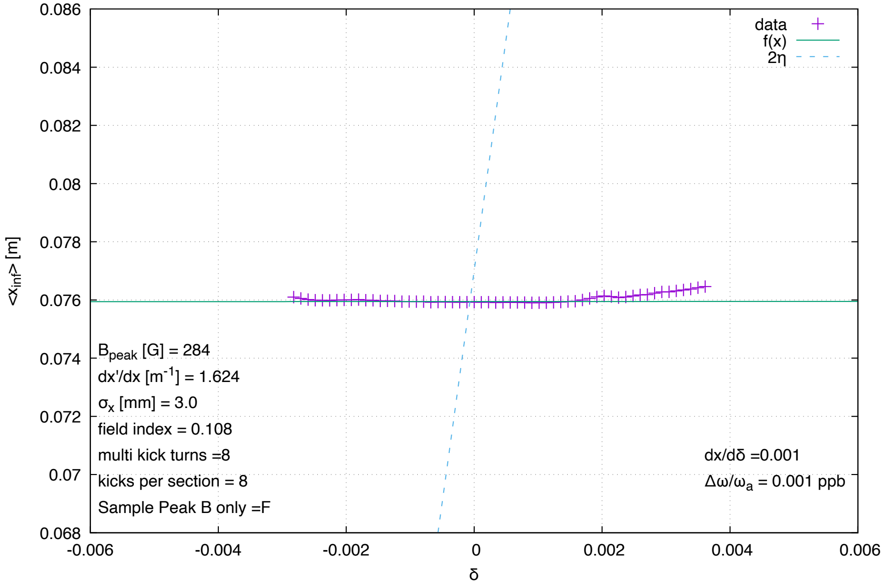


If

- $\sigma_x = 3$ mm
- dx'/dx = 1.624
- Index =0.108
- Kick on multiple turns
- B_{peak} =284 G – measured pulse shape



$$\frac{\Delta\omega}{\omega} = 0.001 \text{ ppb}$$



Conclusion:

$$\langle \Delta\omega_a \rangle = \frac{d\langle \phi_a \rangle}{dx_{inf}} \frac{d\langle x_{inf} \rangle}{d\delta} \frac{1}{\gamma} \frac{d\langle \gamma \rangle}{dt}$$

Momentum acceptance

$$\rightarrow -\frac{1}{2}\left(A - (x_{inf} - \overline{\theta_k}\beta) - \frac{(\overline{x'}_{inf}\beta)^2}{A + (x_{inf} - \overline{\theta_k}\beta)}\right) < \eta\delta < \frac{1}{2}\left(A + (x_{inf} - \overline{\theta_k}\beta) - \frac{(\overline{x'}_{inf}\beta)^2}{A - (x_{inf} - \overline{\theta_k}\beta)}\right)$$

=> $$\frac{d\langle x_{inf} \rangle}{d\delta} \lesssim 1\mathrm{m}$$

$$\frac{\Delta\omega_a}{\omega_a} \lesssim 1[\mathrm{ppb}]$$

for any plausible injected distribution

Details of the final calculation

$$\langle \Delta\omega_a \rangle = \frac{d\langle\phi_a\rangle}{d\gamma}\frac{d\langle\gamma\rangle}{dt} = \frac{d\langle\phi_a\rangle}{dx_{inf}}\frac{dx_{inf}}{d\gamma}\frac{d\langle\gamma\rangle}{dt}$$

$$\frac{d\langle\phi_a\rangle}{dx_{M5}} = -0.251 \text{ m/rad} \qquad \frac{d\langle\phi_a\rangle}{dx'_{M5}} = 0.771$$

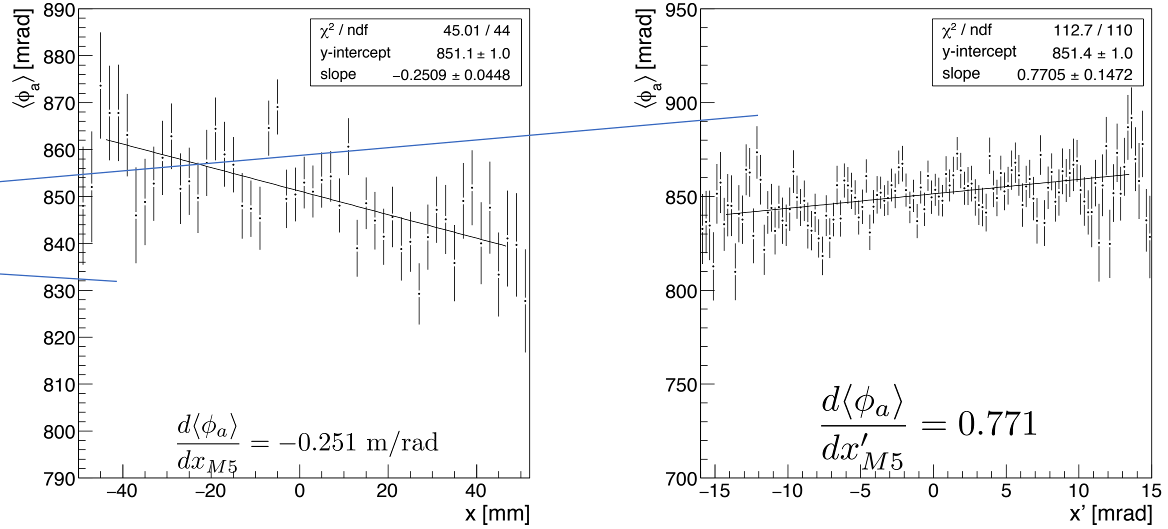


FIG. 1: MC simulation of average z-x spin angle at end of M5[2]

Use BMAD to propagate from the end of M5 to inflector exit to get $\frac{dx_{M5}}{dx_{inf}} = -0.1057$

Then $\frac{d\langle\phi_a\rangle}{dx_{inf}} = \left(\frac{d\langle\phi_a\rangle}{dx_{M5}}\right)\left(\frac{dx_{M5}}{dx_{inf}}\right)$

$$= (-0.25 \text{ mrad/mm}) \times (\frac{1}{-0.1057}) = 2.37 \text{ mrad/mm}$$

Next $\frac{dx_{inf}}{d\gamma} = \frac{dx_{inf}}{d\delta}\frac{1}{\gamma}$ and from doc-db 3477 $\frac{d\langle\gamma\rangle}{dt} = 5.6 \times 10^{-7}\frac{1}{\mu s}.$

$$\Delta\omega_a = \left(2.37\frac{\text{rad}}{\text{m}}\right)\frac{1}{29.3}\left(5.6 \times 10^{-7}\frac{1}{\mu s}\right)\frac{d\langle x_{inf}\rangle}{d\delta}[\text{m}] = 4.5 \times 10^{-8}\ m_\delta\ [\text{m}]$$

=> $\frac{\Delta\omega_a}{\omega_a} = 1.1\ m_\delta\ [\text{ppb}]$

$$\langle \Delta\omega_a \rangle = \frac{d\langle\phi_a\rangle}{d\gamma}\frac{d\langle\gamma\rangle}{dt} = \frac{d\langle\phi_a\rangle}{dx_{inf}}\frac{dx_{inf}}{d\gamma}\frac{d\langle\gamma\rangle}{dt}$$

$$\frac{d\langle\phi_a\rangle}{dx_{M5}} = -0.251 \text{ m/rad}$$

$$\frac{d\langle\phi_a\rangle}{dx'_{M5}} = 0.771$$

Propagate from the end of M5 to inflector exit

$$\frac{d\langle\phi_a\rangle}{dx_{inf}} = \left(\frac{d\langle\phi_a\rangle}{dx_{M5}}\right)\left(\frac{dx_{M5}}{dx_{inf}}\right)$$

$$\frac{dx_{M5}}{dx_{inf}} = -\frac{1}{0.1057}$$

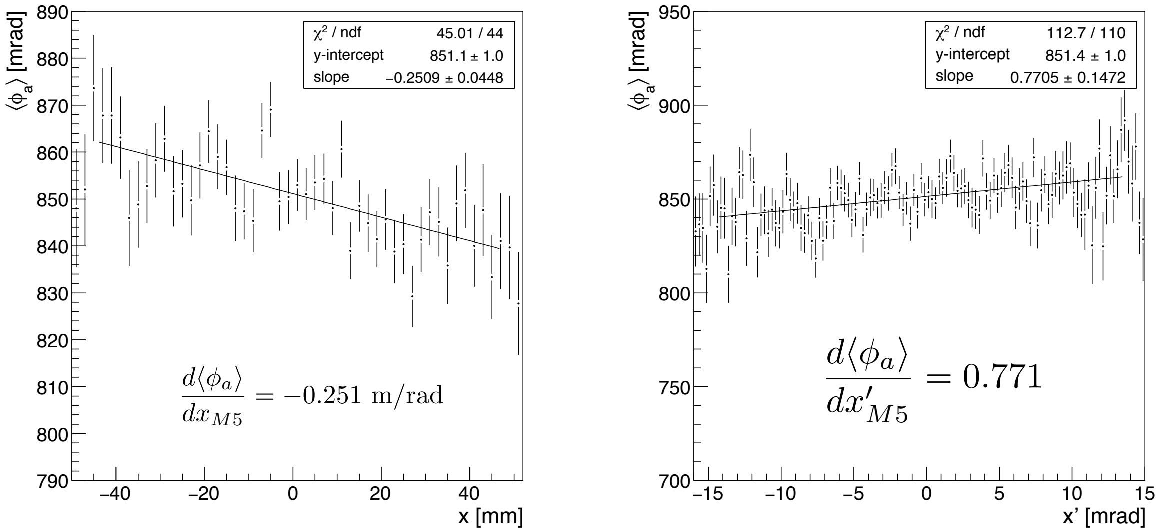


FIG. 1: MC simulation of average z-x spin angle at end of M5[2]

$$\frac{d\langle\phi_a\rangle}{dx_{inf}} = \left(\frac{d\langle\phi_a\rangle}{dx_{M5}}\right)\left(\frac{dx_{M5}}{dx_{inf}}\right)$$
$$= (-0.25 \text{ mrad/mm}) \times (\frac{1}{-0.1057}) = 2.37 \text{ mrad/mm}$$

$$\langle\Delta\omega_a\rangle = \frac{d\langle\phi_a\rangle}{dx_{inf}}\frac{dx_{inf}}{d\delta}\frac{d\delta}{d\gamma}\frac{d\langle\gamma\rangle}{dt}$$
$$= (2.37 \text{ mrad/mm})\frac{dx_{inf}}{d\delta}\frac{1}{\gamma}\frac{d\langle\gamma\rangle}{dt}$$

$$\frac{d\delta}{d\gamma} = \frac{d(\gamma-\gamma_0)}{\gamma_0} = \frac{1}{\gamma_0}$$

$$\frac{d\langle\gamma\rangle}{dt} = 5.6\times 10^{-7}\frac{1}{\mu s}.$$

From doc-db 3477 for E821

$$\Delta\omega = (2.37 \text{ mrad/mm})\left(\frac{dx_{inf}}{d\delta}[\text{mm}]\right)\frac{5.6\times 10^{-7}}{29.3}\frac{1}{\mu s}$$
$$= 0.45\times 10^{-7}\frac{\text{mrad}}{\mu s - \text{mm}}\left(\frac{dx_{inf}}{d\delta}[\text{mm}]\right)$$

Momentum acceptance in terms of initial conditions $x_{inf}, x'_{inf}, \delta, n$ and the kick θ_k

$$\rightarrow -\frac{1}{2}\left(A-(x_{inf}-\overline{\theta_k}\beta)-\frac{(\overline{x'}_{inf}\beta)^2}{A+(x_{inf}-\overline{\theta_k}\beta)}\right) < \eta\delta < \frac{1}{2}\left(A+(x_{inf}-\overline{\theta_k}\beta)-\frac{(\overline{x'}_{inf}\beta)^2}{A-(x_{inf}-\overline{\theta_k}\beta)}\right)$$

$$\left\{ \begin{array}{l} \overline{x'_{inf}} \equiv x'_{inf}+\theta_k\cos(Q_x\frac{\pi}{2}) \\ \overline{\theta_k} \equiv \theta_k\sin(Q_x\frac{\pi}{2}) \end{array} \right.$$

A *is the collimator aperture*

Average

$$\langle\eta\delta\rangle = \frac{1}{2}(x_{inf}-\theta_k\beta)\left(1-\frac{(x'_{inf}\beta)^2}{A^2-(x_{inf}-\theta_k\beta)^2}\right)$$

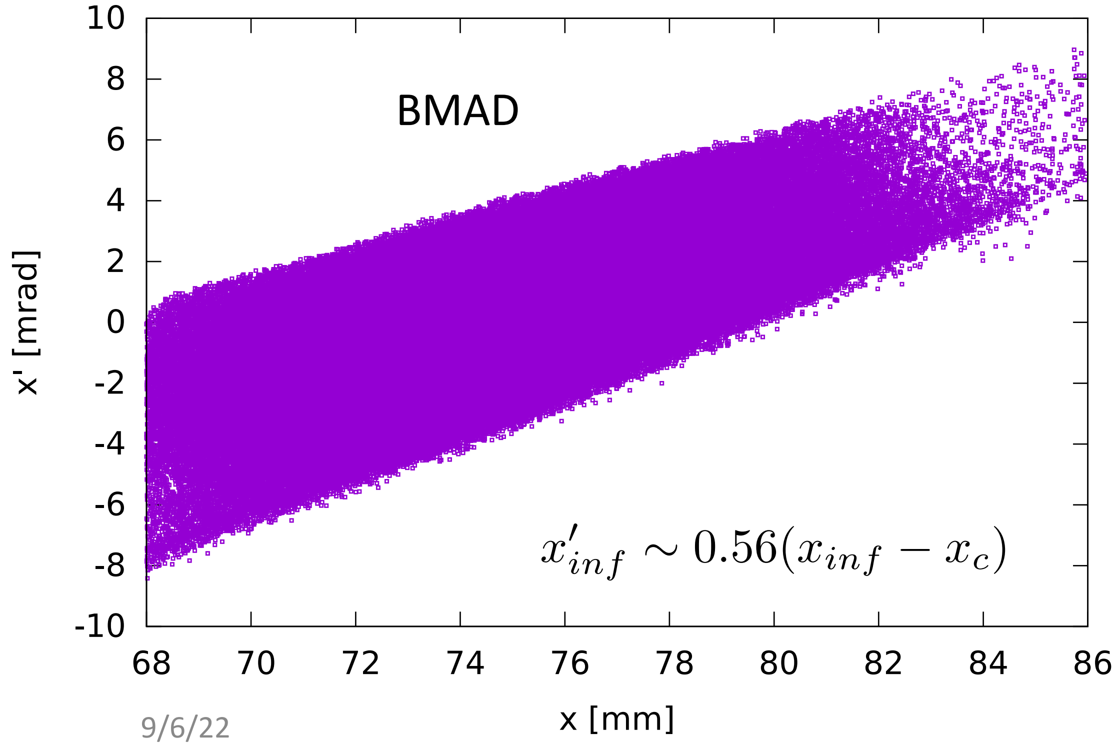
If $x'_{inf} = 0$

$$\langle\eta\delta\rangle = \frac{1}{2}(x_{inf}-\theta_k\beta)$$

$$\frac{d\delta}{dx_{inf}} = \frac{1}{2\eta} \Rightarrow \frac{dx_{inf}}{d\delta} = 2\eta$$

Possible explanations

- Extracting x-x’ correlation from spin phase vs x,x’ is flawed
- Spin phase vs x,x’ plots are for some place other than the end of M5 (our assumption when propagating to the inflector exit)
- Neglecting the width of the x’ distribution is a poor assumption



Phase space at inflector exit