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Pitch Systematic

The conventional strategy for estimating the correction due to the vertical component of the muon velocity is based on measurement of the distribution in the vertical phse space. The vertical velocity component is then related to the measured vertical distribution, and that the velocity component is proportional to the average of $(\tilde{\beta} \cdot \tilde{\mathbf{B}})\tilde{\beta}$ along the trajectory of the muon. We test the methodology in simulation by comparing the shift in ω_a based on $\langle(\tilde{\beta} \cdot \tilde{\mathbf{B}})\tilde{\beta}\rangle$ with spin tracking.

Thomas Equation

According to the Thomas Equation (Jackson 11.170) time dependence of the spin is related to electromagnetic fields, and particle momentum, charge and mass, according to

$$\frac{d\mathbf{s}}{dt} = \frac{e}{mc}\mathbf{s}\times\left[\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)\mathbf{B} - \left(\frac{g}{2} - 1\right)\frac{\gamma}{\gamma+1}(\tilde{\beta}\cdot\mathbf{B})\beta - \left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\beta\times\mathbf{E}\right] \tag{1}$$

The equation of motion for the muon is (Jackson 11.168)

$$\frac{d\tilde{\beta}}{dt} = \frac{e}{mc\gamma}\left[\mathbf{E} + \tilde{\beta}\times\mathbf{B} - (\tilde{\beta}\cdot\mathbf{E})\tilde{\beta}\right] \tag{2}$$

We measure $\frac{d(\hat{\beta}\cdot\mathbf{s})}{dt}$.

$$\frac{(d\hat{\beta}\cdot\mathbf{s})}{dt} = \hat{\beta}\cdot\frac{d\mathbf{s}}{dt} + \frac{1}{\beta}[\mathbf{s} - (\hat{\beta}\cdot\mathbf{s})\hat{\beta}]\cdot\frac{d\beta}{t} \tag{3}$$

The above are combined to give (Jackson 11.171)

$$\frac{d(\hat{\beta}\cdot\mathbf{s})}{dt} = -\frac{e}{mc}\mathbf{s}_\perp\cdot\left[\left(\frac{g}{2} - 1\right)(\hat{\beta}\times\mathbf{B}) + \left(\frac{g}{2} - 1 - \frac{1}{\gamma^2 - 1}\right)\mathbf{E}\right] \tag{4}$$

Let $\hat{\beta}\cdot\mathbf{s} = s\cos\theta$, and $\mathbf{s}_\perp = \hat{\beta}\times\mathbf{s} = s\sin\theta\hat{\mathbf{s}}_\perp$ so that

$$\frac{d(\hat{\beta}\cdot\mathbf{s})}{dt} = -s\sin\theta\frac{d\theta}{dt} = -\frac{e}{mc}\mathbf{s}_\perp\cdot\left[\left(\frac{g}{2} - 1\right)(\hat{\beta}\times\mathbf{B}) + \left(\frac{g}{2} - 1 - \frac{1}{\gamma^2 - 1}\right)\mathbf{E}\right] \tag{5}$$

$$\rightarrow -\omega = -\frac{e}{mc}\hat{\mathbf{s}}_\perp\cdot\left[\left(\frac{g}{2} - 1\right)(\hat{\beta}\times\mathbf{B}) + \left(\frac{g}{2} - 1 - \frac{1}{\gamma^2 - 1}\right)\mathbf{E}\right] \tag{6}$$

$$\rightarrow -\frac{e}{mc}\hat{\mathbf{s}}_\perp\cdot\left[\left(\frac{g}{2} - 1\right)(\hat{\beta}\times\mathbf{B}) + \left(\frac{g}{2} - 1 - \frac{1}{\gamma^2 - 1}\right)\hat{\beta}\times\mathbf{E}\right] \tag{7}$$

$$\tag{8}$$

That last step since $\mathbf{s}_\perp\cdot\mathbf{E} = \hat{\beta}\times\mathbf{s}\cdot\mathbf{E} = \mathbf{s}\cdot(\hat{\beta}\times\mathbf{E}) = \mathbf{s}_\perp\cdot(\hat{\beta}\times\mathbf{E})$. If $\hat{\beta} = \beta(\cos\psi\hat{\phi} + \sin\psi\hat{z})$ then

$$\omega \rightarrow \omega\frac{(\hat{\beta}\times\mathbf{B})}{B} = \omega\cos\psi \sim \omega(1 - \frac{1}{2}\psi^2) \rightarrow \Delta\omega = \frac{1}{2}\psi^2 \tag{9}$$

Let's try another representation. Assume

$$\vec{\beta} = \beta(\hat{\mathbf{k}}\cos(\psi_0\cos\omega_v t) + \hat{\mathbf{j}}\sin(\psi_0\sin\omega_v t))$$

where ω_v is the vertical betatron frequency and ψ is the pitch angle. Define θ so that $\hat{\beta} \cdot \mathbf{s} = s\cos\theta$ and $\hat{\beta} \times \mathbf{s} = s\sin\theta\hat{\mathbf{s}}_\perp$. Then $\hat{\mathbf{s}}_\perp \cdot \mathbf{B} = B\cos\psi$. ($B$ in the vertical direction). Now

$$
\begin{aligned}
\frac{d(\hat{\beta}\cdot\mathbf{s})}{dt} &= -\frac{e}{mc}\mathbf{s}_\perp \cdot \left[\left(\frac{g}{2}-1\right)(\hat{\beta}\times\mathbf{B}) + \left(\frac{g}{2}-1-\frac{1}{\gamma^2-1}\right)\mathbf{E}\right] \\
\frac{sd(\cos\theta)}{dt} &= -\frac{e}{mc}\mathbf{B}\cdot\left(\frac{g}{2}-1\right)(\mathbf{s}\times\hat{\beta}) \\
-\sin\theta\frac{d\theta}{dt} &= -\frac{e}{mc}B\left(\frac{g}{2}-1\right)\cos\psi(s\sin\theta) \\
\rightarrow \omega_a &= -\frac{e}{mc}B\left(\frac{g}{2}-1\right)\cos\psi
\end{aligned}
$$

Then $\beta \times \mathbf{B} = \hat{\mathbf{i}}\psi B\cos\omega_v t$

$\hat{\beta} = (\cos\alpha\hat{\mathbf{k}} + \sin\alpha\hat{\mathbf{j}})$. Note that $\mathbf{s}_\perp \cdot (\hat{\beta}\times\mathbf{B}) = \mathbf{s}\cdot(\hat{\beta}\times\mathbf{B})$. Then 5 becomes

$$
\frac{d(\hat{\beta}\cdot\mathbf{s})}{dt} = \frac{d}{dt}\left(s(\sin\alpha\sin\theta\sin\phi + \cos\alpha\cos\theta)\right) \sim \frac{d}{dt}\left(s(\cos\alpha\cos\theta)\right) \tag{10}
$$

The pitch angle α oscillates with the vertical betatron frequency which is much greater than the precession frequency with average value of zero. Then

$$
\begin{aligned}
\frac{d(\hat{\beta}\cdot\mathbf{s})}{dt} &= -s(\cos\alpha\sin\theta)\frac{d\theta}{dt} = sB\sin\theta\cos\phi\cos\alpha\left(\frac{g}{2}-1\right) \\
&\rightarrow \quad \omega = -B\cos\phi\left(\frac{g}{2}-1\right)
\end{aligned}
$$

0.0.1 E-field correction

If $\mathbf{E} = E_x\hat{\mathbf{i}}$ and $\tilde{\beta} = \beta\hat{\mathbf{z}}$ then from 5

$$
\frac{d(\hat{\beta}\cdot\mathbf{s})}{dt} = \frac{d}{dt}s(\cos\theta) \tag{11}
$$

Then

$$
\begin{aligned}
\frac{d(\hat{\beta}\cdot\mathbf{s})}{dt} &= -s(\sin\theta)\frac{d\theta}{dt} = sE\sin\theta\cos\phi\left(\frac{g}{2}-1-\frac{1}{\gamma^2-1}\right) \\
&\rightarrow \quad \Delta\omega = -E\cos\phi\left(\frac{g}{2}-1-\frac{1}{\gamma^2-1}\right)
\end{aligned}
$$

Bibliography

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