Betatron contribution to the E-field correction

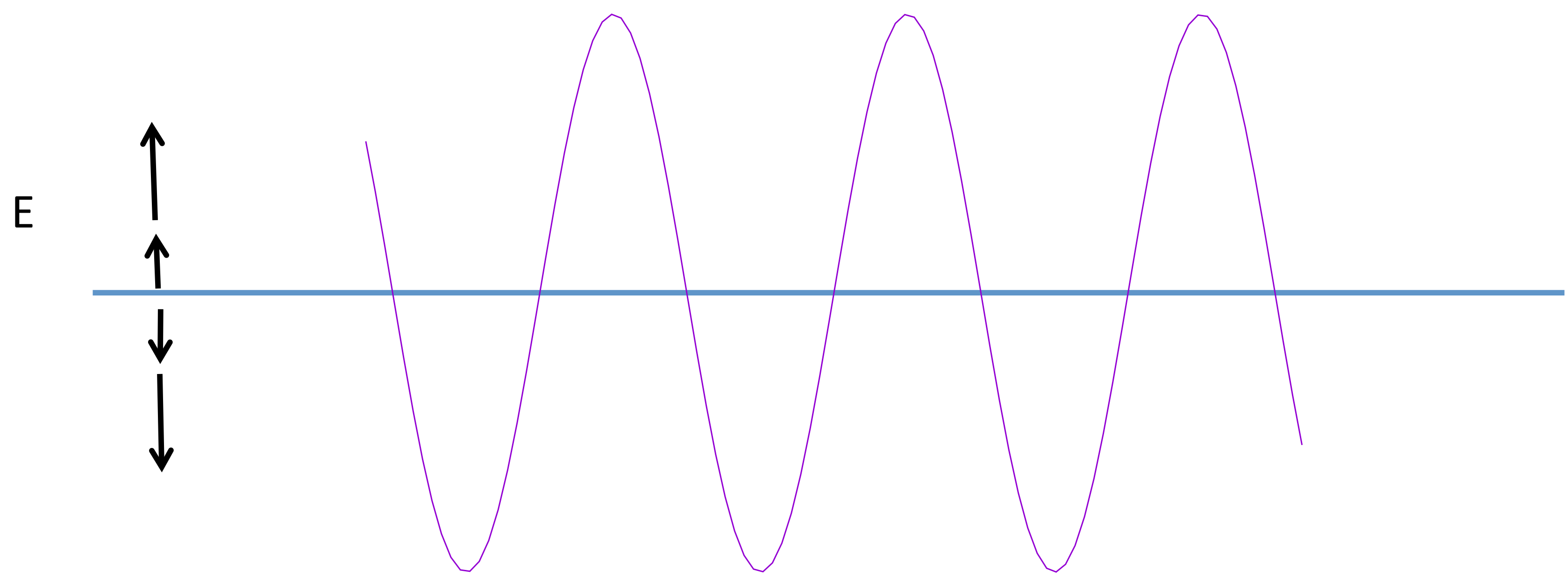
D. Rubin

June 6, 2018

$$\vec{\omega}_a = -\frac{q}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{m^2}{p^2}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right]$$

E-field contribution

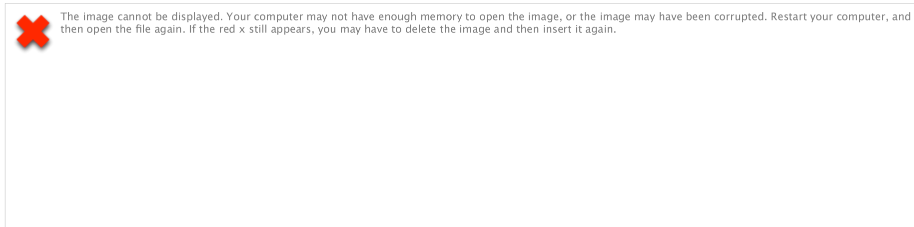
In an ideal cartesian geometry and quad field where the radial field is antisymmetric about the magic radius, the E-field correction is independent of betatron amplitude

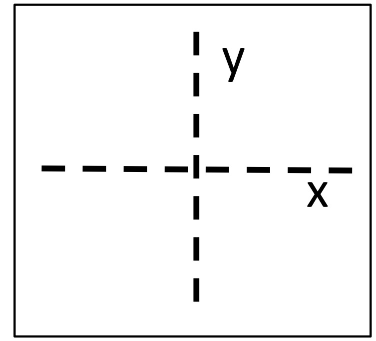


In a curved geometry, the integrated E-field along the trajectory depends on betatron amplitude in two ways

1. Sextupole component of the quads (This component is symmetric about magic radius)
2. Path length (asymmetric about magic radius)

Consider the Laplacian in 2 dimensions and cartesian coordinates





A solution that corresponds to the perfect quadrupole is

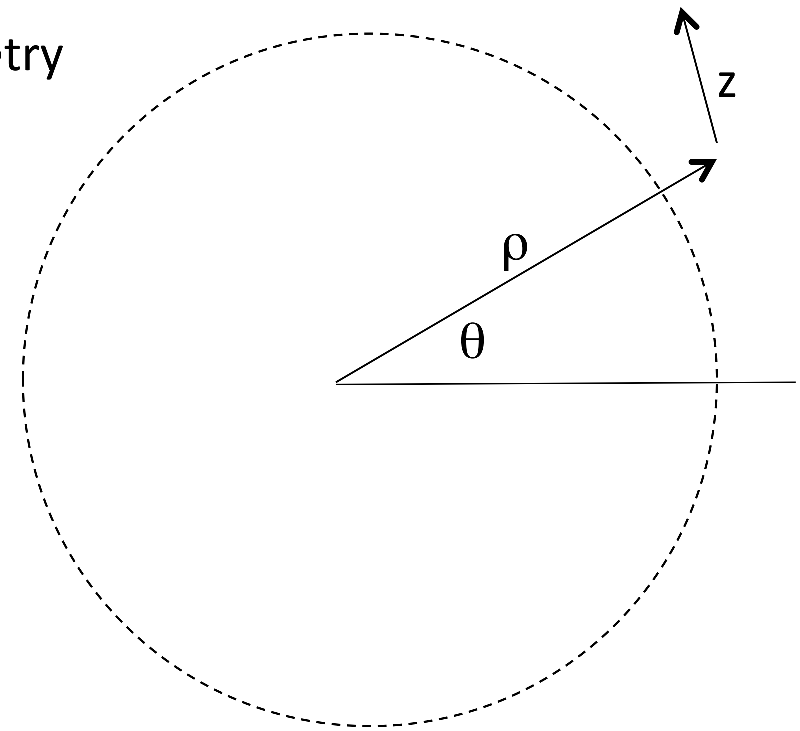
$$V(x,y) = \frac{1}{2}k(x^2 - y^2)$$

Then the divergence gives the E-field, linear in x and y.
and anti symmetric about x,y=0

$$\mathbf{E} = k(x\hat{\mathbf{i}} - y\hat{\mathbf{j}})$$

But the quad plates are curved.
We assumed in the above that there is no z-dependence.
And that is true only in the limit of $\rho \rightarrow \infty$

Cylindrical coordinates are a better match to our geometry
ρ – radial, z – vertical, θ - azimuthal



Laplacian in cylindrical coordinates

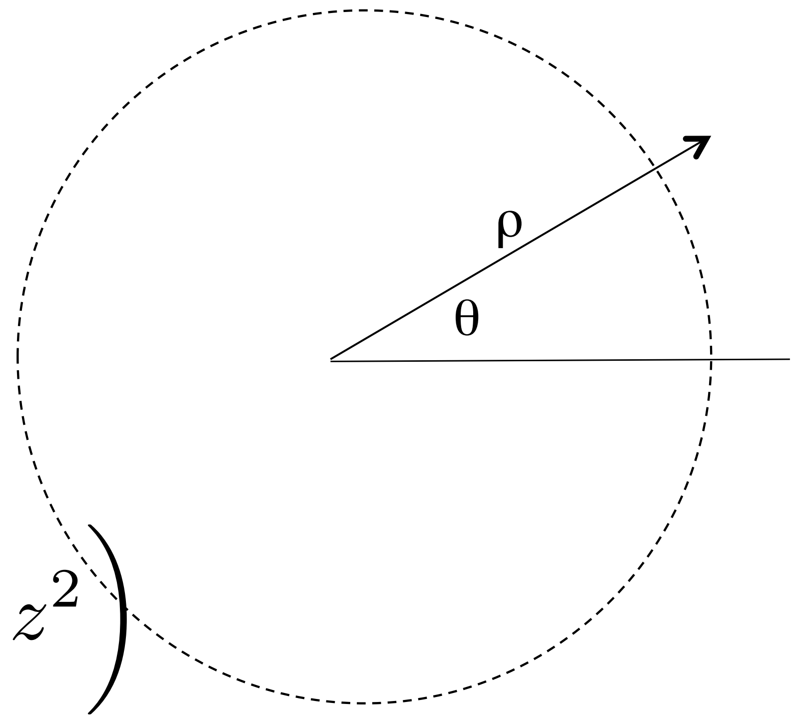
$$\nabla^2 V = \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) V$$

Define x $\rho = \rho_0 + x$

The simple symmetric quadratic potential is **not** a solution to the this Laplacian

$$V(x,z) \neq \frac{1}{2}k(x^2 - z^2)$$

The simplest (lowest order) solution to the 2 D cylindrical Laplacian is



$$V(\rho, z) = k\left(\frac{1}{2}(\rho^2 - 1) - \rho_0 \ln\frac{\rho}{\rho_0} - z^2\right)$$

Then with the substitution

$$\rho = \rho_0 + x$$

And expanding in the limit where $x \ll \rho_0$

$$\nabla V = \mathbf{E} \sim k\left((x - \frac{x^2}{\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}}\right)$$

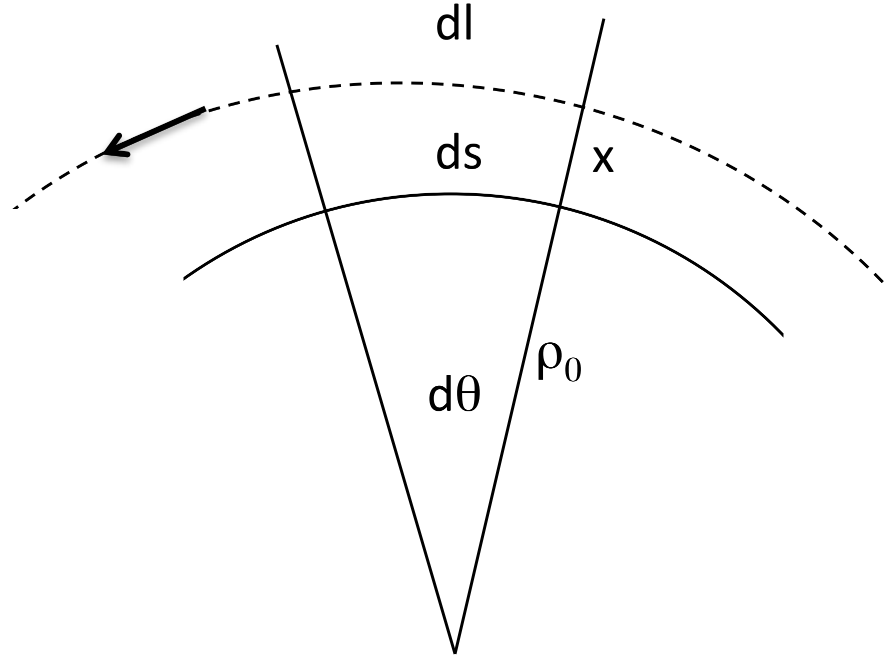
$$\nabla V = \mathbf{E} \sim k\left((x - \frac{x^2}{\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}} \right)$$

Linear term Sextupole

The solution is not unique.
It is possible to find a solution that is linear in x,
but then it is necessarily nonlinear in z (vertical)

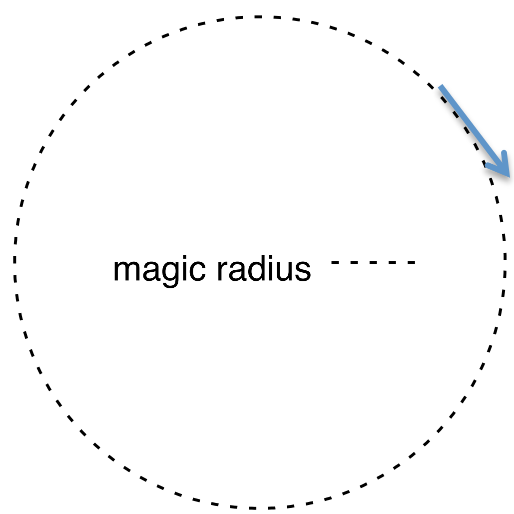
There is inevitably a sextupole component with curved plates independent of the plate shape details and alignment.

Path length

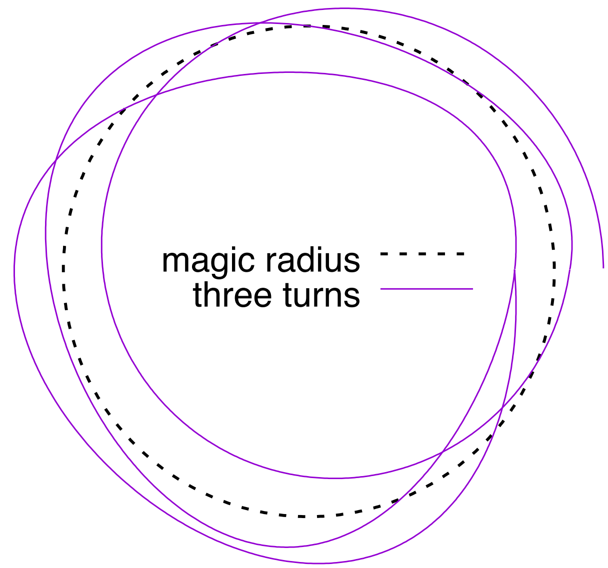


A particle oscillating about the magic radius (ρ_0) spends more time at x>0 than x<0

$$dl = (\rho_0 + x)d\theta$$



The E-field along the trajectory at the magic radius (momentum = p_0) is zero.



But what about the muon with momentum p_0 that oscillates about the magic radius with some betatron amplitude x_β ?

Or the muon with momentum $p_0 + \Delta p$ and betatron amplitude x_β?

$$x = \eta\delta + x_\beta \qquad \delta = \Delta p / p_0$$

$$\nabla V = \mathbf{E} \sim k\left((x - \frac{x^2}{\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}} \right)$$

$$\begin{aligned}
\langle E_r(s) \rangle &= k\langle\left(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2\right)\rangle \\
&= \frac{k}{L}\int_0 \left(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2\right) dl \\
&= \frac{k}{L}\int_0 \left(\eta\delta + x_\beta - \frac{1}{2\rho_0}(\eta\delta + x_\beta)^2\right) (1 + x_\beta/\rho_0)ds
\end{aligned}$$

sextupole

Path length

The average E-field for a muon with momentum $p_0 + \Delta p$ and betatron amplitude x_β is

$$\langle E_r \rangle = k \left(\eta\delta + \frac{1}{2\rho_0}((\eta\delta)^2 + \frac{1}{2}x_{\beta 0}^2) + \dots \right)$$

$$(x_e = \eta\delta)$$

E-field correction

$$C_e = \left(1 - \frac{1}{a_\mu}\frac{m^2}{p^2}\right)\frac{\beta E_r}{cB} \sim \left(1 - \frac{m^2}{a_\mu p_0^2}(1 - 2\frac{\Delta p}{p})\right)\frac{\beta E_r}{cB} \tag{1}$$

Magic momentum $m^2/p_0^2 = a_\mu$

$$x_e = \eta\delta$$

$$C_e(\delta, x_{\beta 0}) \quad \approx \quad 2\frac{\beta k}{cB}\left(\frac{x_e^2}{\eta} + \frac{1}{2\rho_0}(\frac{x_e^3}{\eta} + \frac{1}{2}x_{\beta 0}^2\frac{x_e}{\eta})\right)$$

If $\langle x_e\rangle = \langle\delta\rangle\eta = 0$ then correction is independent of x_β

$$\langle C_e(\delta, x_{\beta 0})\rangle \quad = \quad 2\frac{\beta k}{cB}\eta\langle\delta^2\rangle$$

If $\langle x_e \rangle = \langle \delta \rangle \eta \neq 0$ then according to the Miller/Nguyen rule

To minimize the E-field correction choose p_0 so that

$$2a_\mu \langle \frac{p - p_0}{p_0} \rangle = \frac{m^2}{p_0^2} - a_\mu$$

Then

$$\langle C_e \rangle \sim 2 \left[-\eta(\langle \delta \rangle^2 - \langle \delta^2 \rangle) + \frac{1}{4\rho_0}(\langle \delta \rangle^2 \langle x_{\beta 0}^2 \rangle \right] \frac{\beta k}{cB}$$

Contribution from betatron amplitude