sextupole-like component of quadrupoles

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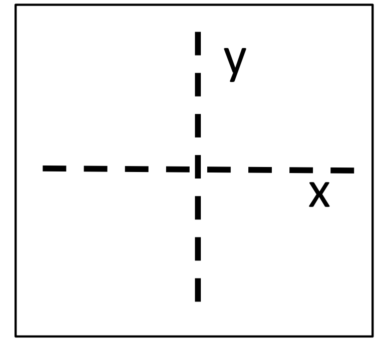
Consider the Laplacian in 2 dimensions and cartesian coordinates

$$\nabla^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) V(x, y)$$



A solution that corresponds to the perfect quadrupole is

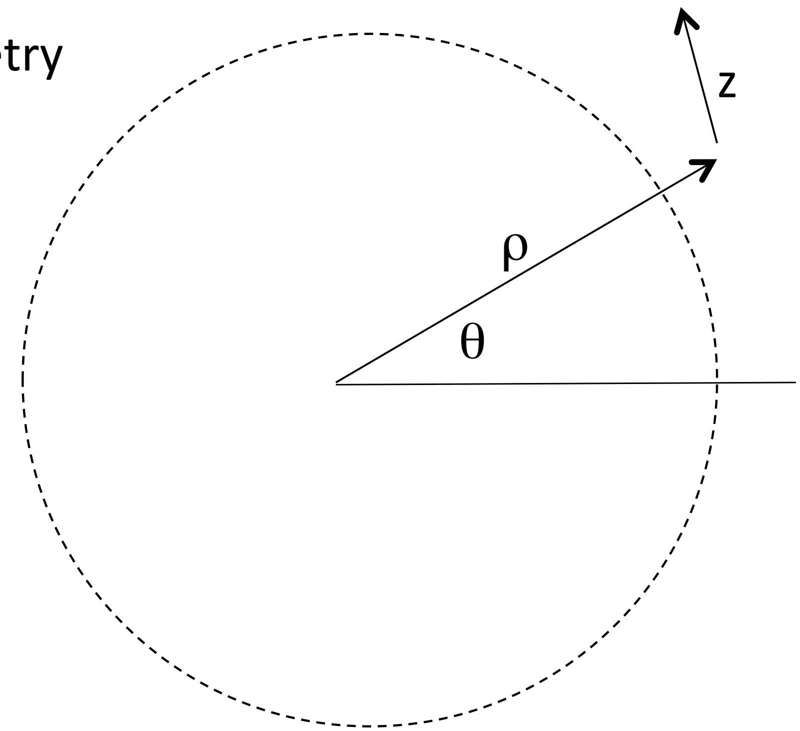
$$V(x, y) = \frac{1}{2} k (x^2 - y^2)$$

Then the divergence gives the E-field, linear in x and y.
and anti symmetric about x,y=0

$$\mathbf{E} = k(x\hat{\mathbf{i}} - y\hat{\mathbf{j}})$$

But the quad plates are curved.
We assumed in the above that there is no z-dependence.
And that is true only in the limit of $\rho \rightarrow \infty$

Cylindrical coordinates are a better match to our geometry
ρ – radial, z – vertical, θ - azimuthal



Laplacian in cylindrical coordinates

$$\nabla^2 V = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right) V$$

Define x $$\rho = \rho_0 + x$$

The simple symmetric quadratic potential is **not** a solution to this Laplacian

$$V(x,z) \neq \frac{1}{2}k(x^2 - z^2)$$

Cartesian coordinates

$$E_x - iE_y = (b_n - ia_n)\frac{(x+iy)^n}{r_0^n}$$

Satisfies Maxwell for any n

Cartesian coordinates

$$E_x - iE_y = (b_n - ia_n)\frac{(x+iy)^n}{r_0^n}$$

Cylindrical

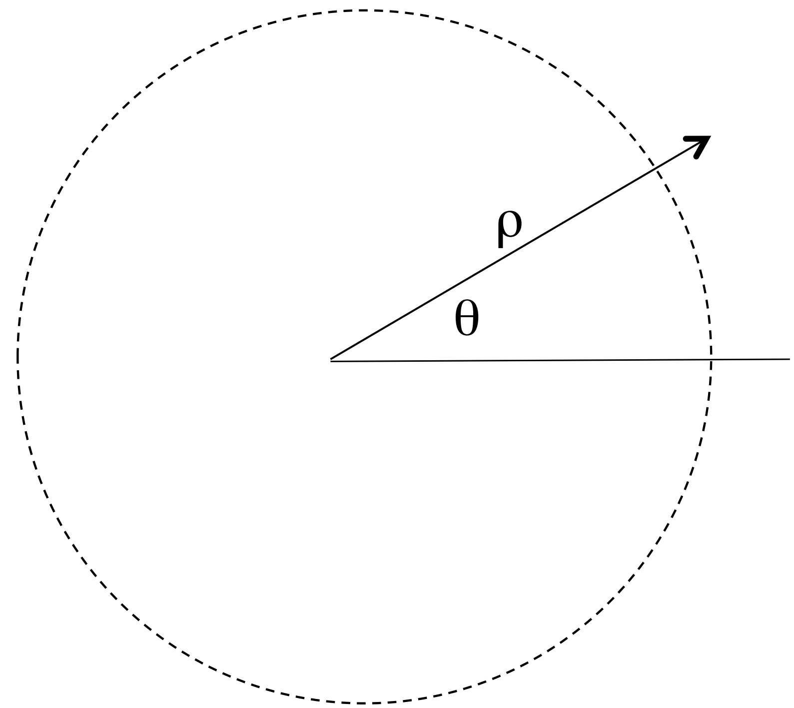
$$\mathbf{E} = -\sum_{n=0}^{\infty} \rho^n \left(a_n \tilde{\nabla}\phi_n^i + b_n \tilde{\nabla}\phi_n^r\right)$$

$$\phi_n^r = \frac{-1}{1+n} \sum_{p=0}^{((n+1)/2)} \binom{n+1}{2p} (-1)^p F_{n+1-2p}(\tilde{r})\tilde{y}^{2p}$$

$$\phi_n^i = \frac{-1}{1+n} \sum_{p=0}^{(n/2)} \binom{n+1}{2p+1} (-1)^p F_{n-2p}(\tilde{r})\tilde{y}^{2p+1}$$

$$\begin{aligned} F_1 &= \ln\tilde{r} \\ F_2 &= \frac{1}{2}(\tilde{r}^2 - 1) - \ln\tilde{r} \\ F_3 &= \frac{3}{2}[-(\tilde{r}^2 - 1) + (\tilde{r}^2 + 1)\ln\tilde{r}] \end{aligned}$$

The simplest (lowest order) solution to the 2 D cylindrical Laplacian is



$$V(\rho, z) = k\left(\frac{1}{2}(\frac{\rho^2}{\rho_0^2} - 1) - \ln\frac{\rho}{\rho_0} - \left(\frac{z}{\rho_0}\right)^2\right)$$

Then with the substitution

$$\rho = \rho_0 + x$$

And expanding in the limit where $x \ll \rho_0$

$$\nabla \mathbf{V} = E \sim k\left((x - \frac{x^2}{2\rho_0} + \ldots)\hat{\rho} - z\hat{\mathbf{z}}\right)$$

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Linear term

Sextupole-like

The solution is not unique.
It is possible to find a solution that is linear in x,
but then it is necessarily nonlinear in z (vertical)

There is inevitably a sextupole component with curved plates independent of the plate shape details and alignment.

Cartesian coordinates – no z- dependence

$$E_x - iE_y = (b_n - ia_n)\frac{(x+iy)^n}{r_0^n}$$

Satisfies Maxwell for any n

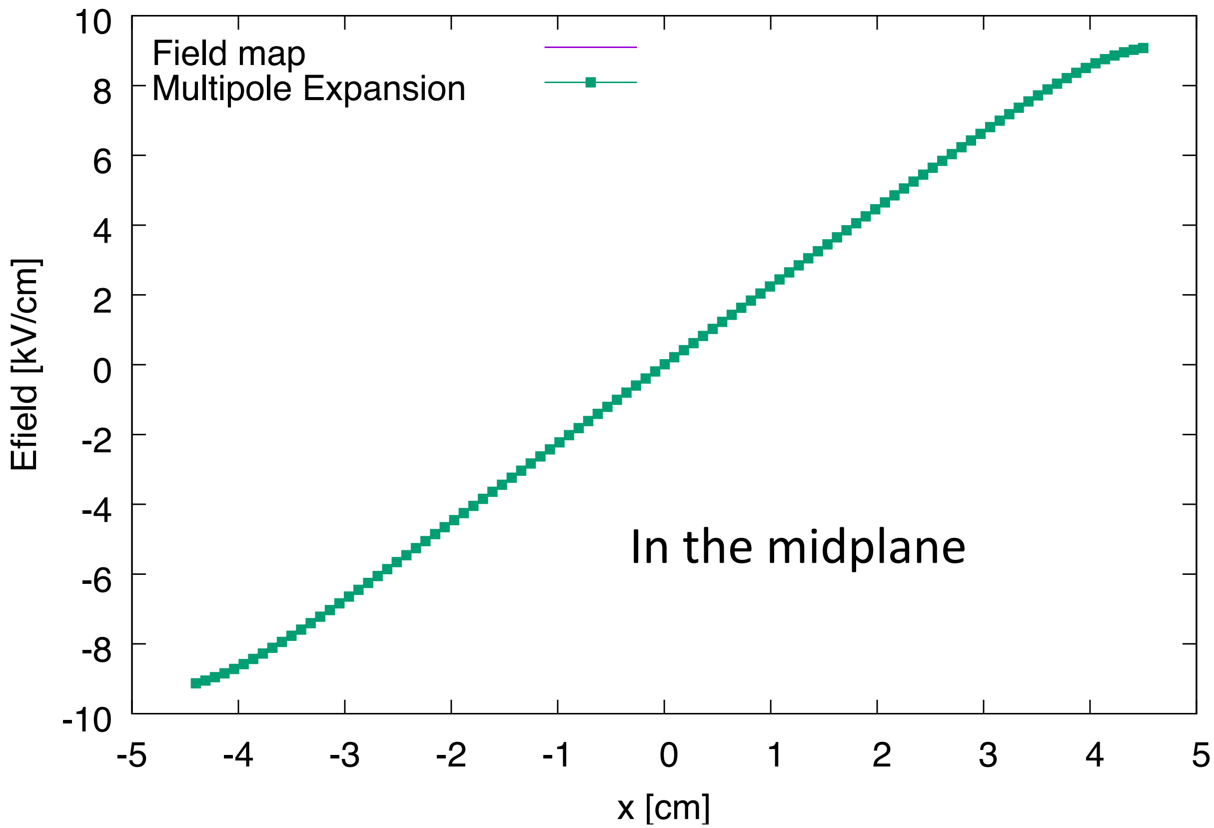
Not so in cylindrical coordinates. But we can expand as

$$E_x = \sum_n b_n \left(\frac{x}{r_0}\right)^n$$

In the midplane (y=0)

$$E_x = \sum_n b_n \left(\frac{x}{r_0}\right)^n$$

Fit to Wanwei's 3-D Opera field map



Multipole	Normal (b_n)	Skew (a_n)
1	1.01609E+06	-1.19899E+00
2	-2.71281E+03	7.71402E+00
3	-1.45524E+04	-1.50954E+01
4	-6.90244E+02	1.46521E+00
5	-5.23865E+03	4.35213E+01
6	1.00671E+02	-4.83666E+01
7	1.21107E+03	7.75800E+01
8	-1.43120E+02	4.99963E+00
9	-9.02621E+04	5.20048E+00
10	2.87638E+02	-3.47144E+01
11	5.36519E+03	3.43053E+01
12	1.05747E+02	2.39598E+00
13	8.00742E+02	-9.14390E+00

This was our guess

$$E \sim k\left((x - \frac{x^2}{2\rho_0} + \dots)\hat{\rho} - z\hat{\mathbf{z}} \right)$$

General form

$$E_x = \sum_n \frac{b_n}{r_0^n} x^n$$

$$\Rightarrow \frac{b_2/r_0^2}{b_1/r_0} = -\frac{1}{2\rho_0} = -0.0703\mathrm{m}^{-1}$$

The fitted values give

$$\Rightarrow \frac{b_2/r_0^2}{b_1/r_0} = -0.0593\mathrm{m}^{-1}$$