Decoherence of Betatron Motion

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For each time slice we determine the minimum δ_{min} and maximum δ_{max} captured momenta. If we assume that particle exit the inflector straight ahead and that there is 90 degrees phase advance to the kicker, the betatron amplitude (including the sign) for each momentum withing that range is

$$a(\delta) = (\delta_{max} + \delta_{min} - \delta)\eta$$

The betatron motion downstrem of the kicker is given by

$$a(\delta, t) = a(\delta)\sin((\omega + \Delta\omega(\delta))t + \phi(\delta))$$

If we assume that the betatron frequency depends linearly on momentum, $\omega(\delta) = (1 - b\delta)\omega_0$, where we leave b for later, Then the collective motion of all momenta for the slice is

$$\begin{aligned}
\frac{1}{\delta_{max} - \delta_{min}} \int_{\delta_{min}}^{\delta_{max}} a(\delta, t) d\delta \; &= \; \frac{\eta}{\delta_{max} - \delta_{min}} \int_{\delta_{min}}^{\delta_{max}} (\delta_{max} + \delta_{min} - \delta) \sin(\omega(1 - b\delta)t) d\delta \\
&\rightarrow \eta \int_{\delta_{min}}^{\delta_{max}} (\delta_{max} + \delta_{min}) \sin(\omega(1 - b\delta)t) d\delta - \eta \int_{\delta_{min}}^{\delta_{max}} \delta \sin(\omega(1 - b\delta)t) d\delta \\
&= \eta \int_{\delta_{min}}^{\delta_{max}} (\delta_{max} + \delta_{min}) \sin(\omega(1 - b\delta)t) d\delta - \eta \int_{\delta_{min}}^{\delta_{max}} \delta \left(\sin(\omega t)\cos(\omega b\delta t) - \cos(\omega t)\sin(\omega b\delta t)\right) d\delta
\end{aligned}$$

Integrate the three terms one at a time
First term

$$\begin{aligned}
\eta \int_{\delta_{min}}^{\delta_{max}} (\delta_{max} + \delta_{min}) \sin(\omega(1 - b\delta)t) d\delta &= \eta \int_{\delta_{min}}^{\delta_{max}} (\delta_{max} + \delta_{min}) \sin(xt) \frac{dx}{-\omega b} \\
&= \frac{\eta}{\omega b t} (\delta_{max} + \delta_{min}) \cos(\omega(1 - b\delta)t|_{\delta_{min}}^{\delta_{max}} \\
&= -\frac{\eta}{\omega b t} (\delta_{max} + \delta_{min}) \left(\cos(\omega(1 - b\delta_{max})t - \cos(\omega(1 - b\delta_{min})t\right)
\end{aligned}$$

Second term

$$\begin{aligned}
-\eta \int_{\delta_{min}}^{\delta_{max}} \delta \left(\sin(\omega t)\cos(\omega b\delta t)\right) d\delta &= -\eta \sin(\omega t) \int_{\delta_{min}}^{\delta_{max}} \frac{d}{d(\omega b t)} \left(\sin(\omega b\delta t)\right) d\delta \\
&= -\eta \sin(\omega t) \frac{d}{d(\omega b t)} \int_{\delta_{min}}^{\delta_{max}} \left(\sin(\omega b\delta t)\right) d\delta \\
&= \eta \sin(\omega t) \frac{d}{d(\omega b t)} \left(\frac{1}{\omega b t} \cos(\omega b\delta t)\right)_{\delta_{min}}^{\delta_{max}} \\
&= -\eta \sin(\omega t) \left(\frac{1}{(\omega b t)^2} \cos(\omega b\delta t) + \frac{\delta}{\omega b t} \sin(\omega b\delta t)\right)_{\delta_{min}}^{\delta_{max}}
\end{aligned}$$

Third term

$$\begin{aligned}
\eta \int_{\delta_{min}}^{\delta_{max}} \delta \left(\cos(\omega t)\sin(\omega b\delta t)\right) d\delta &= -\eta \cos(\omega t) \int_{\delta_{min}}^{\delta_{max}} \frac{d}{d(\omega b t)} \left(\cos(\omega b\delta t)\right) d\delta \\
&= -\eta \cos(\omega t) \frac{d}{d(\omega b t)} \int_{\delta_{min}}^{\delta_{max}} \left(\cos(\omega b\delta t)\right) d\delta \\
&= -\eta \cos(\omega t) \frac{d}{d(\omega b t)} \left(\frac{1}{\omega b t} \sin(\omega b\delta t)\right)_{\delta_{min}}^{\delta_{max}} \\
&= \eta \cos(\omega t) \left(\frac{1}{(\omega b t)^2} \sin(\omega b\delta t) - \frac{\delta}{\omega b t} \cos(\omega b\delta t)\right)_{\delta_{min}}^{\delta_{max}}
\end{aligned}$$