Momentum acceptance model

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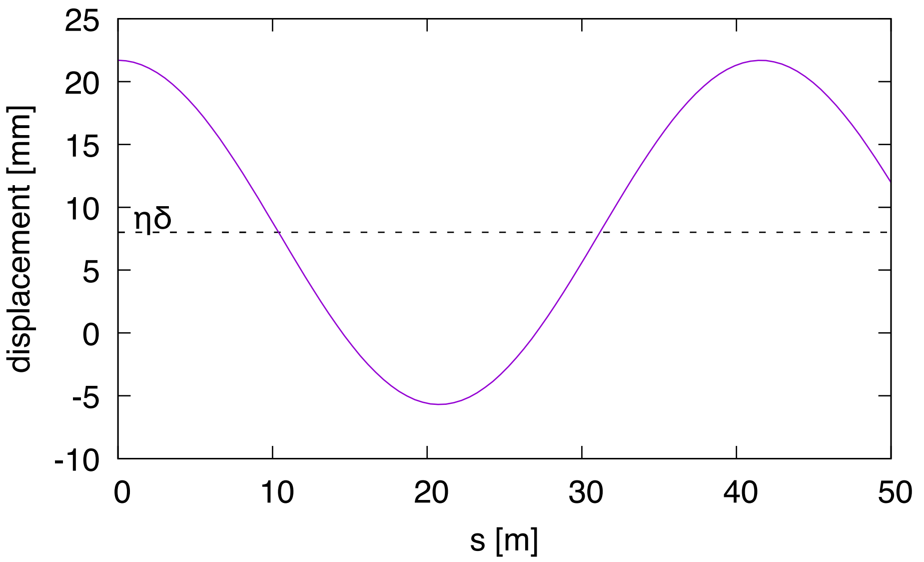
Goal:

Develop an analytic model of momentum acceptance in order to understand origin and dependencies of time-momentum correlation of captured particles

Free betatron motion

displacement $$x(s) = a\sqrt{\beta}\cos(\phi(s) - \phi_0) + \eta\delta$$



With continuous quads

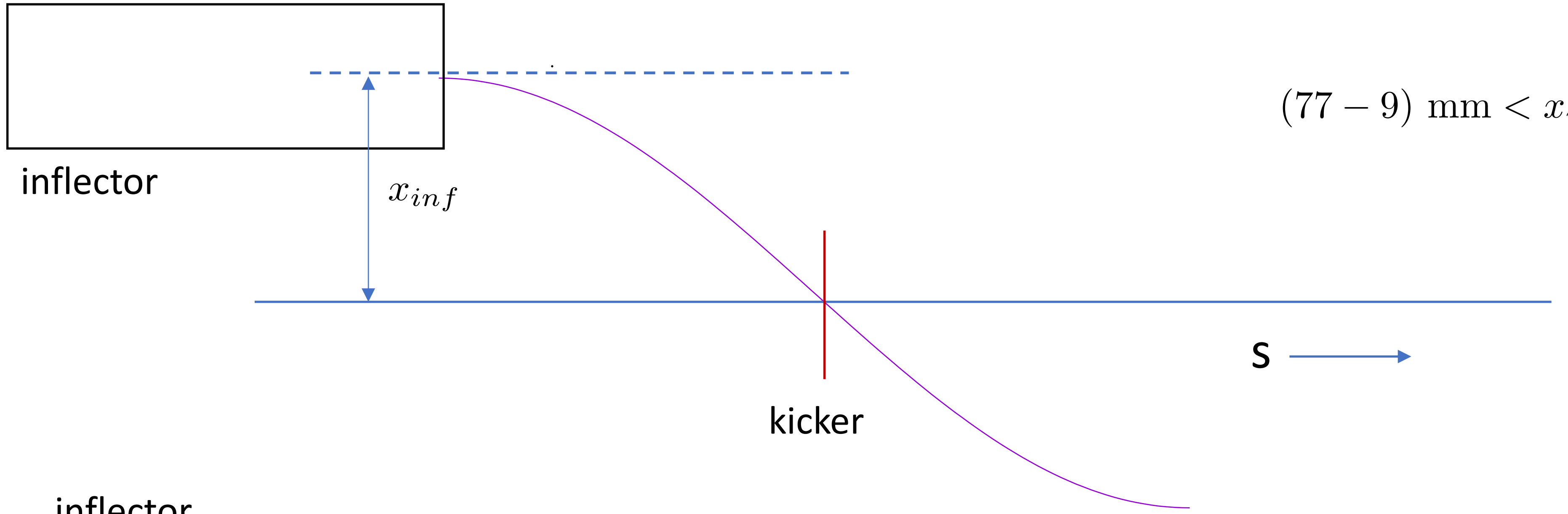
$$\beta = \frac{R_0}{\sqrt{1-n}}, \qquad \eta = \frac{R_0}{1-n}, \qquad Q_x = \sqrt{1-n}$$

Betatron phase $$\phi(s) = Q_x \frac{s}{R_0}, \quad (\rightarrow \phi_{FT} = 2\pi Q_x)$$

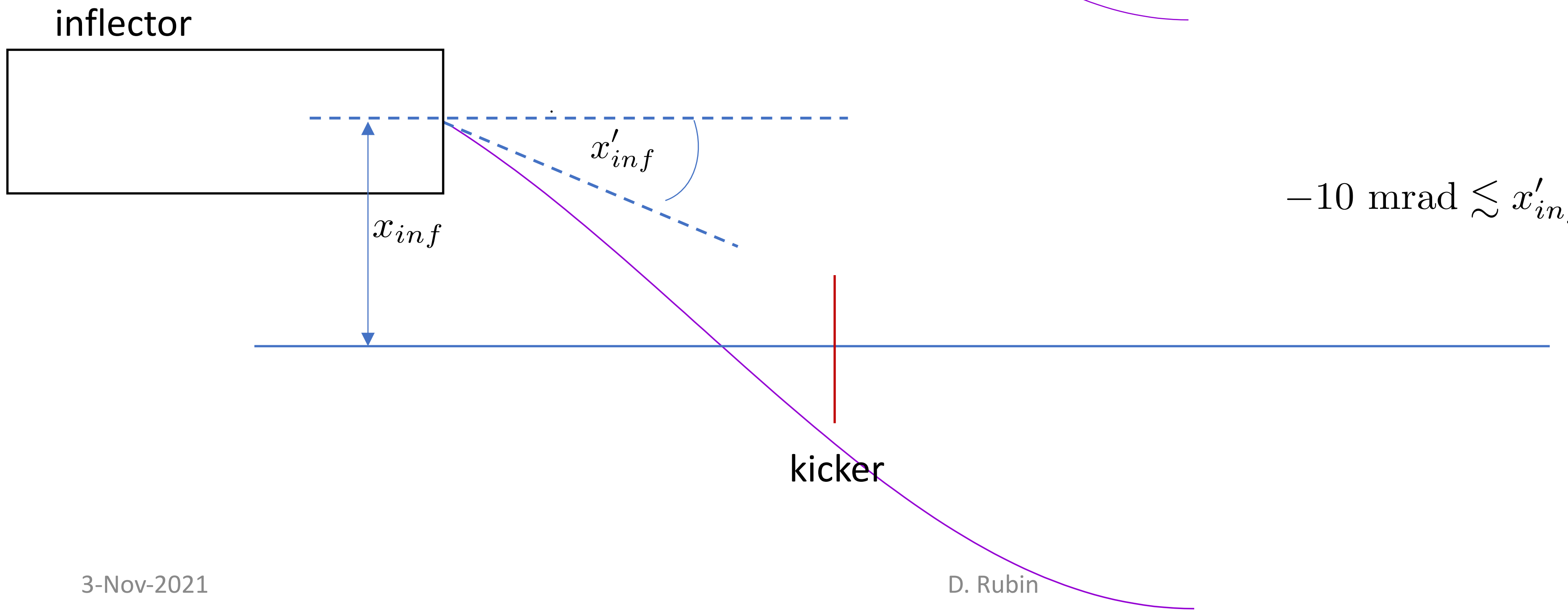
Then the angle of the trajectory:

$$x'(s) = \frac{d}{ds}x(s) = -a\sqrt{\beta}\sin(\phi - \phi_0)\phi' = -\frac{a}{\sqrt{\beta}}\sin(\phi - \phi_0)$$

Some Definitions



$$(77-9)\ \text{mm} < x_{inf} < (77+9)\ \text{mm}$$



$$-10\ \text{mrad} \lesssim x'_{inf} \lesssim 10\ \text{mrad}$$

Trajectory of injected particle

At the inflector exit $(\phi = 0)$

$$x_{inf} = a\sqrt{\beta}\cos\phi_0 + \eta\delta, \qquad x'_{inf} = \frac{a}{\sqrt{\beta}}\sin\phi_0$$

Write initial phase offset in terms of initial x and x′

$$x(s) = a\sqrt{\beta}(\cos(\phi - \phi_0) + \eta\delta = a\sqrt{\beta}[\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0] + \eta\delta$$

$$\Rightarrow x(s) - \eta\delta = (x_{inf} - \eta\delta)\cos\phi + \beta x'_{inf}\sin\phi$$

Trajectory in terms of initial conditions and betatron phase

Invariant amplitude

The betatron amplitude $$a = \frac{1}{\sqrt{\beta}}[(x-\eta\delta)^2 + (\beta x')^2]^{1/2}$$

The betatron amplitude on exit from the inflector

$$a_0 = \frac{1}{\sqrt{\beta}}[(x_{inf}-\eta\delta)^2 + (\beta x'_{inf})^2]^{1/2}$$

Kicker

At the kicker $\phi(s) = \phi_k$

Displacement and angle at the kicker

$$x_k = (x_{inf} - \eta\delta)\cos\phi_k + \beta x'_{inf}\sin\phi_k + \eta\delta$$
$$x'_k = -\frac{1}{\beta}(x_{inf} - \eta\delta)\sin\phi_k + x'_{inf}\cos\phi_k$$

Immediately beyond the kick (imagine an infinitesimally thin kicker)

$$x_{k+} = x_k$$
$$x'_{k+} = x'_k + \theta_k$$

$$x_{k+} = (x_{inf} - \eta\delta)\cos\phi_k + \beta x'_{inf}\sin\phi_k + \eta\delta$$

$$x'_{k+} = -\frac{1}{\beta}(x_{inf} - \eta\delta)\sin\phi_k + x'_{inf}\cos\phi_k + \theta_k$$

Betatron amplitude beyond the kicker

$$a_{k+} = \frac{1}{\sqrt{\beta}}[(x_{k+} - \eta\delta)^2 + \beta^2(x'_{k+})^2]^{1/2}$$

$$a_{k+} = \left[a_0^2 + \beta\left(\theta_k^2 + 2\theta_k\left(-\frac{1}{\beta}(x_{inf} - \eta\delta)\sin\phi_k + x'_{inf}\cos\phi_k\right)\right)\right]^{1/2}$$

The amplitude is a function of $a(x_{inf}, x'_{inf}, \phi_k, \theta_k, \delta)$

The kick and the kick phase can be chosen such that $a_{k+} = 0$

If $x'_{inf} = 0$ and $\phi_k = \pi/2$, the amplitude beyond the kicker is zero if

$$\theta_k^0 = \frac{x_{inf} - \eta\delta}{\beta}$$

If $x'_{inf} \neq 0$, the kick that minimizes amplitude is always greater than θ_k^0

Capture

Beyond the kicker

$$x(s) = \sqrt{\beta} a_{k+} \cos(\phi - \phi_0) + \eta\delta$$

The muon is captured if

$$-A < x \ < \ A$$

Where A is the collimator aperture ($A = 45 \text{ mm}$)

$$x < a_{k+}\sqrt{\beta} + \eta\delta < A$$
$$x > -a_{k+}\sqrt{\beta} + \eta\delta > -A$$

The momentum acceptance for amplitude $a(x_{inf}, x'_{inf}, \phi_k, \theta_k, \delta)$ is

$$-A + a_{k+}\sqrt{\beta} < \eta\delta \ < \ A - a_{k+}\sqrt{\beta}$$

Momentum acceptance

$$-A + a_{k+}\sqrt{\beta} < \eta\delta \ < \ A - a_{k+}\sqrt{\beta}$$

(remember that amplitude a_{k+} depends on momentum)

Rearrange to give the momentum acceptance in terms of initial conditions and kicks

$$\rightarrow -\frac{1}{2}\left(A - (x_{inf} - \overline{\theta_k}\beta) - \frac{(\overline{x'}_{inf}\beta)^2}{A + (x_{inf} - \overline{\theta_k}\beta)}\right) < \eta\delta < \frac{1}{2}\left(A + (x_{inf} - \overline{\theta_k}\beta) - \frac{(\overline{x'}_{inf}\beta)^2}{A - (x_{inf} - \overline{\theta_k}\beta)}\right)$$

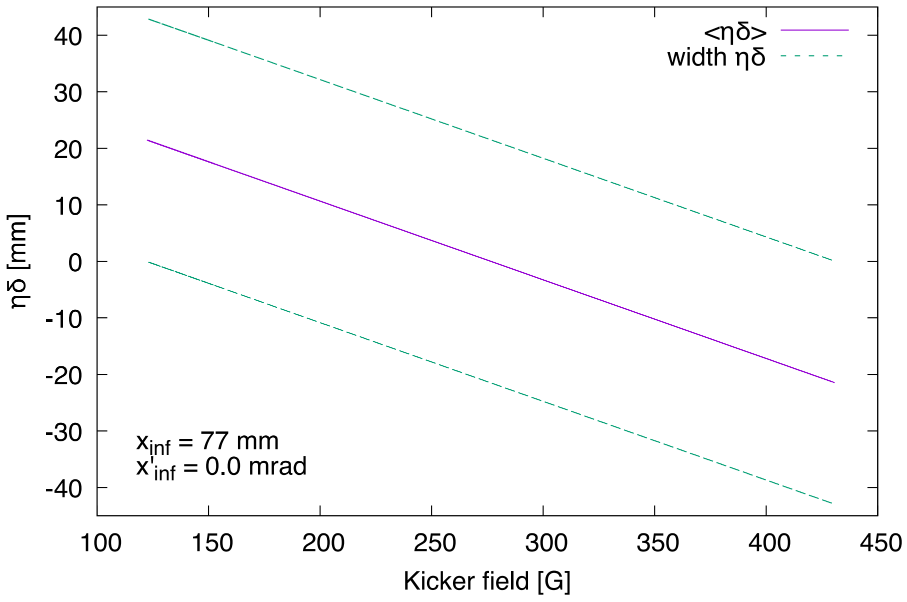
If the injection angle is zero and the kick is chosen to zero a_{k+} for the magic momentum, $\theta_k = \frac{x_{inf}}{\beta}$

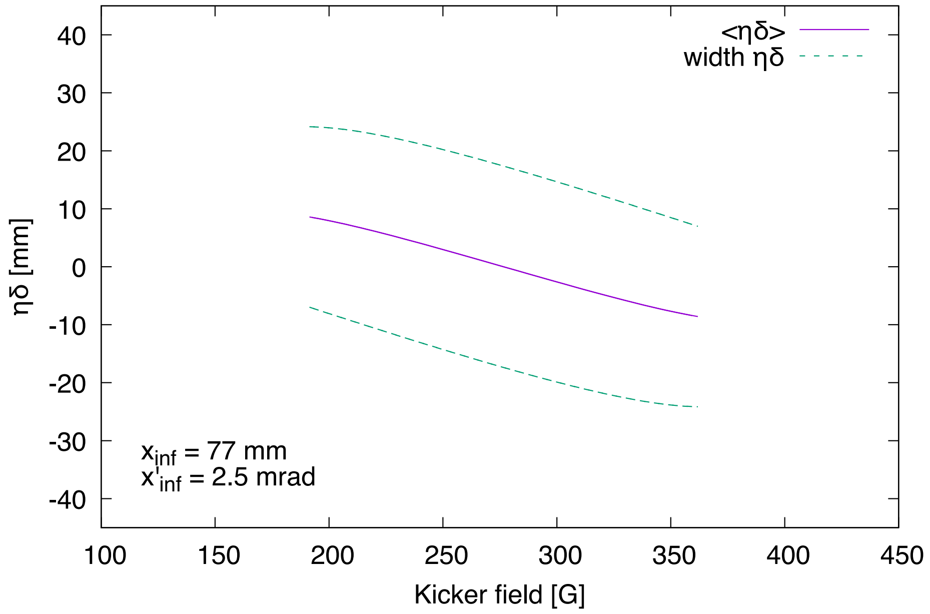
Then

$$-\frac{A}{2} < \eta\delta < \frac{A}{2}$$

injection angle reduces momentum acceptance

(*Here* $\overline{\theta_k} = \theta_k$ *and* $\overline{x'_{inf}} = x'_{inf}$)





Injection angle $x'_{inf} = 0$

The minimum kick captures momenta in the range $0 \leq \eta\delta \leq A$

The maximum kick captures momenta in the range $-A < \eta\delta \leq 0$

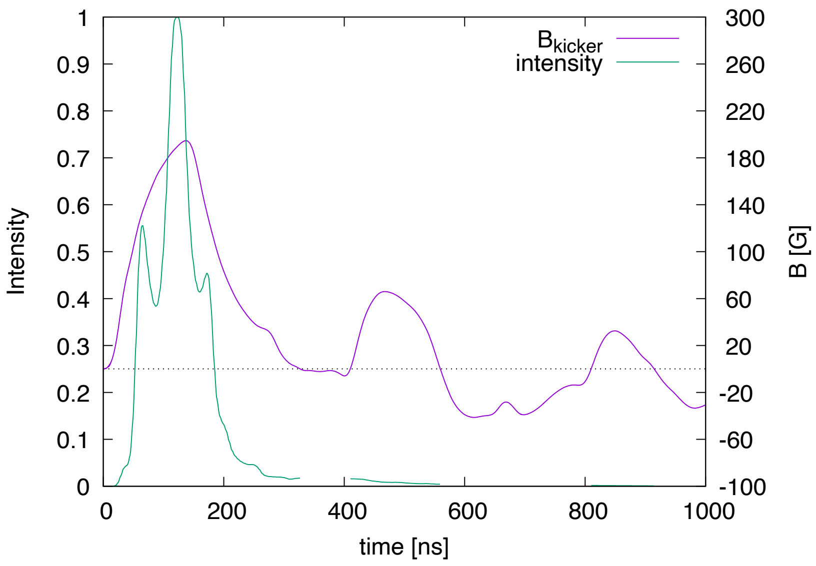
The ideal kick (the kick that steers the magic momentum onto the magic radius) captures momenta in the range $-\frac{A}{2} < \eta\delta < \frac{A}{2}$

Injection angle $x'_{inf} = -2.5 \text{ mrad}$

Injection angle shrinks momentum acceptance

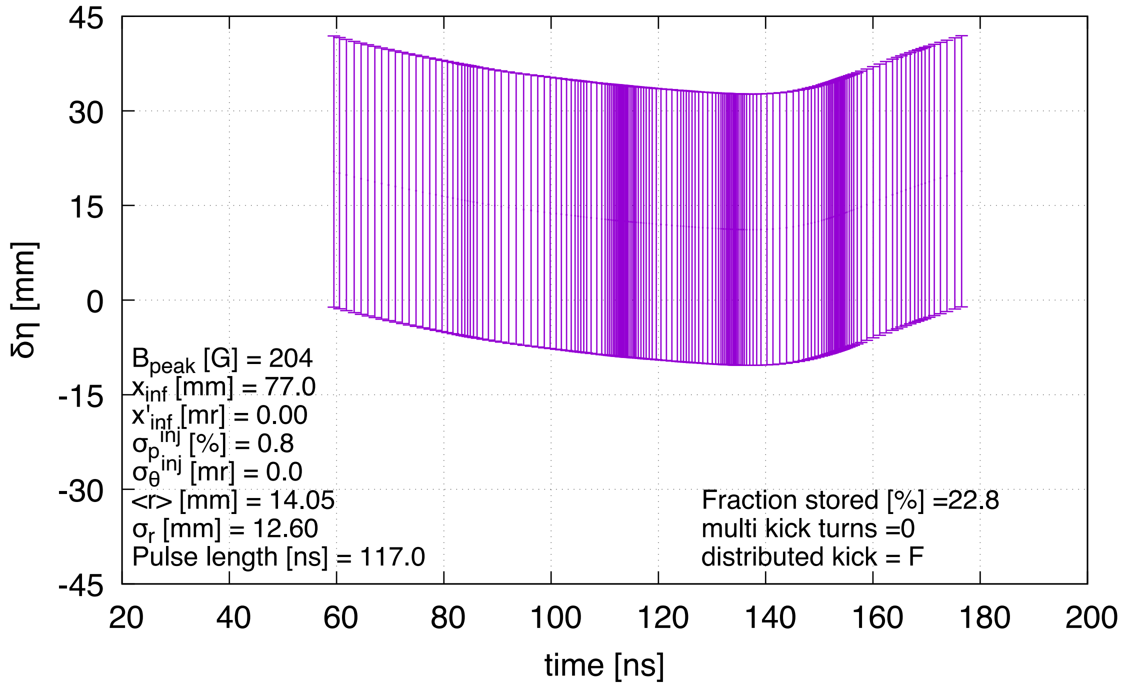
Momentum acceptance vs kick

Our inequality gives us the momentum acceptance as a function of kick



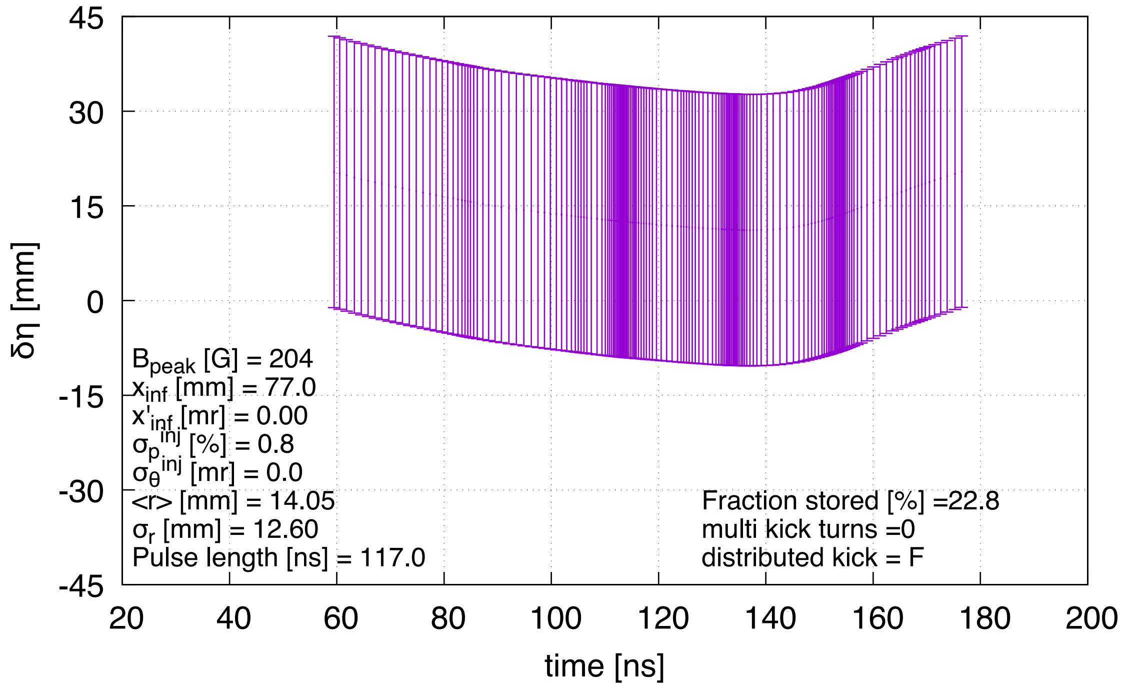
Time dependence of kicker field and injected pulse

Momentum bite versus time of captured particles



Field index = 0

$$\phi_k = \pi/2, \quad x'_{inf} = 0$$

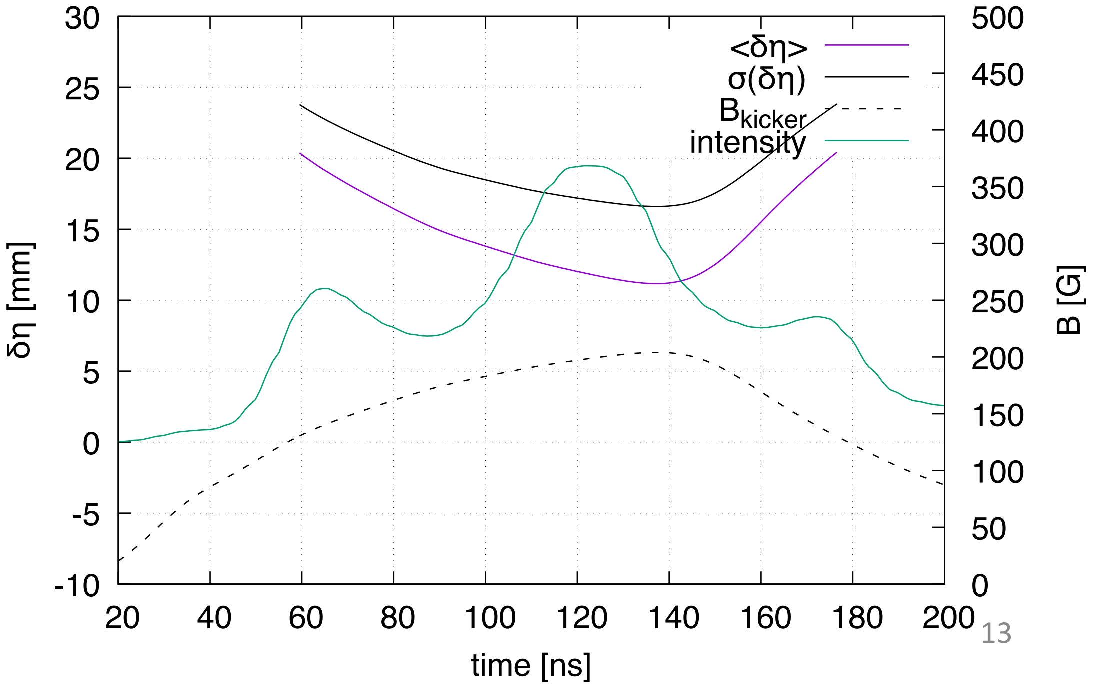


If we assume the momentum distribution is gaussian with $\sigma = 0.8\%$ we can compute the average and width of momentum in each captured slice

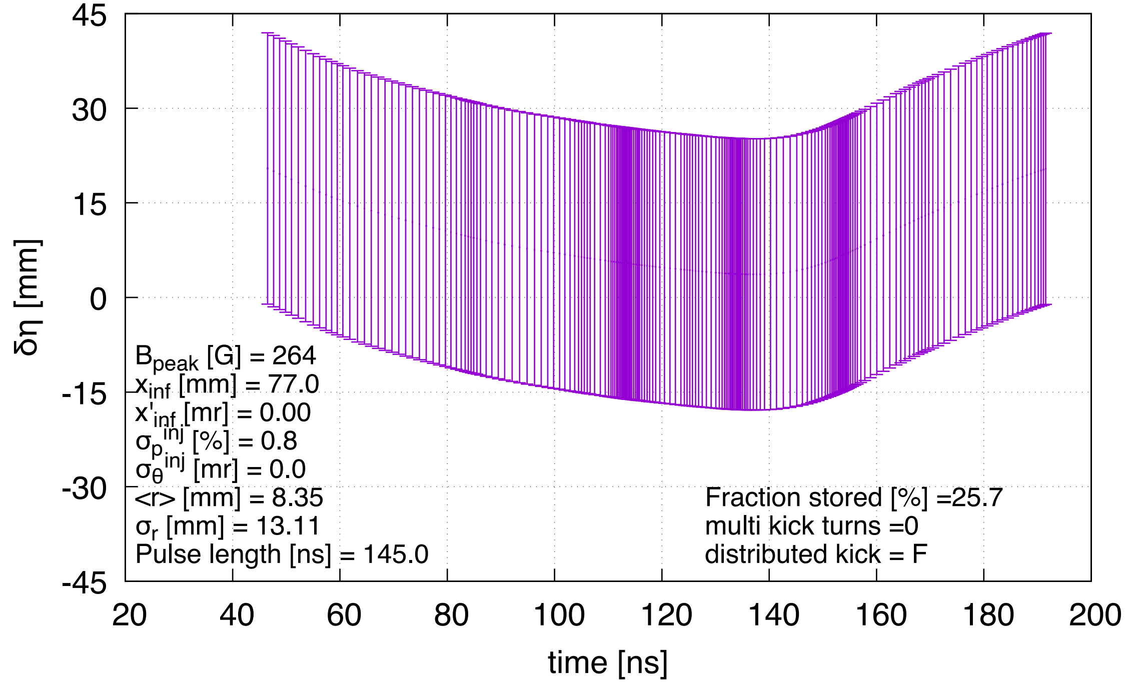
Field index = 0

B_{peak} =204 G

$$\phi_k = \pi/2, \quad x'_{inf} = 0$$



Bigger kick

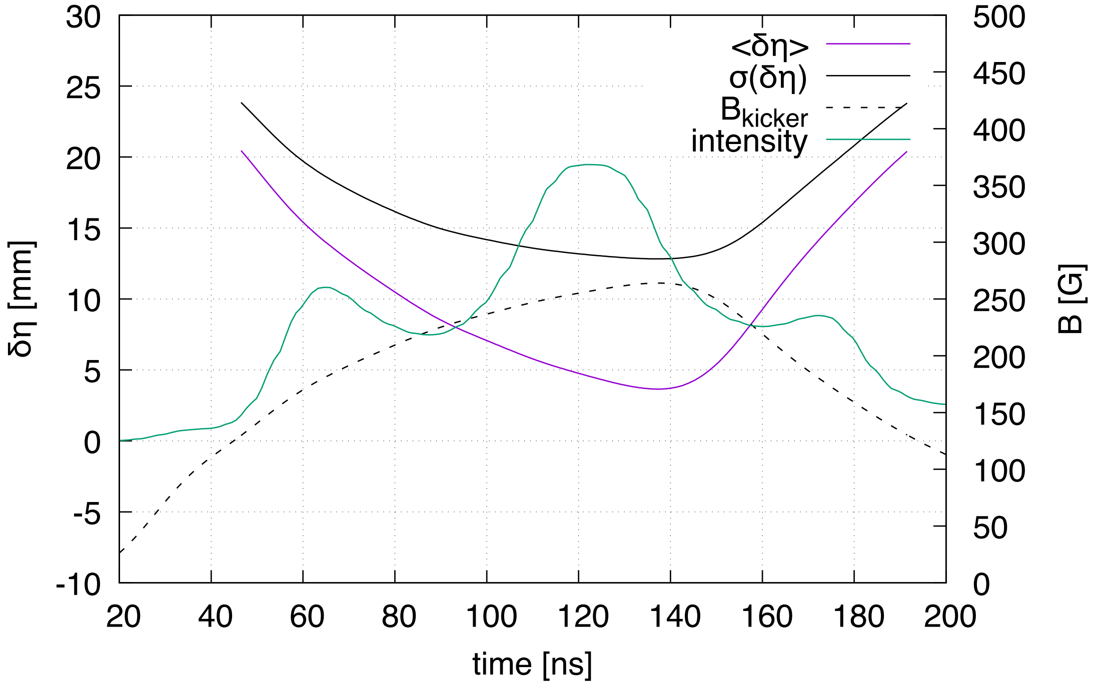


Time momentum correlation increases with higher kicker field.

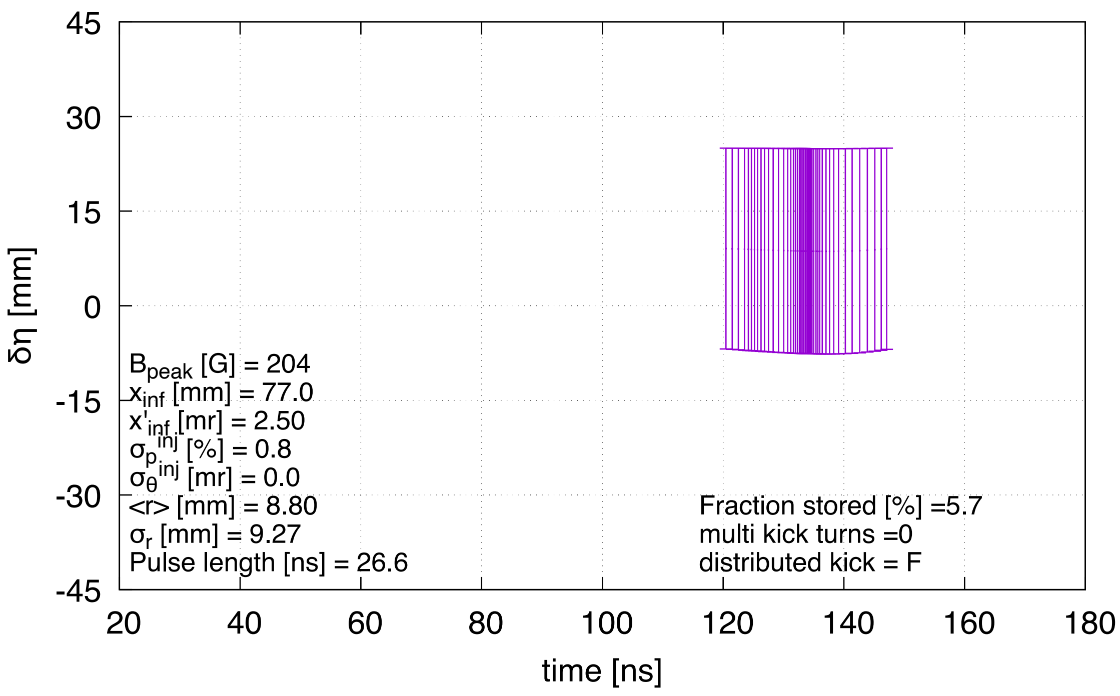
Field index = 0

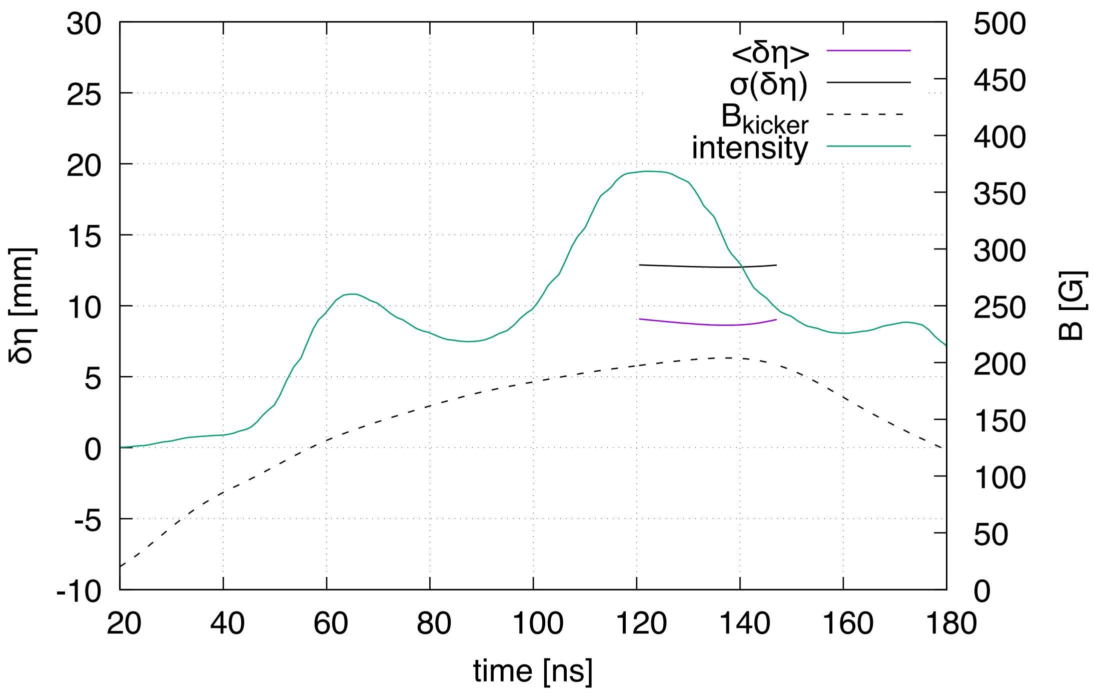
B_{peak} =264 G

$$\phi_k = \pi/2, \quad x'_{inf} = 0$$



Finite injection angle decreases momentum acceptance (and correlation)



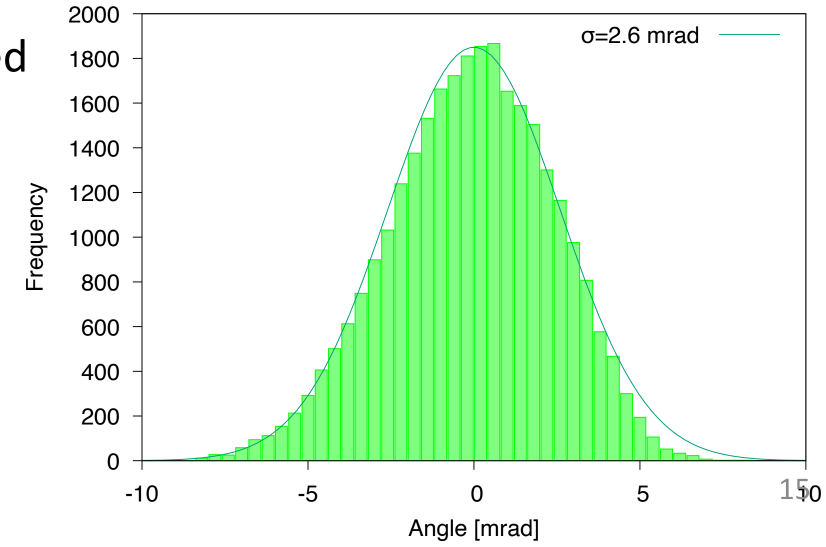


Field index =0

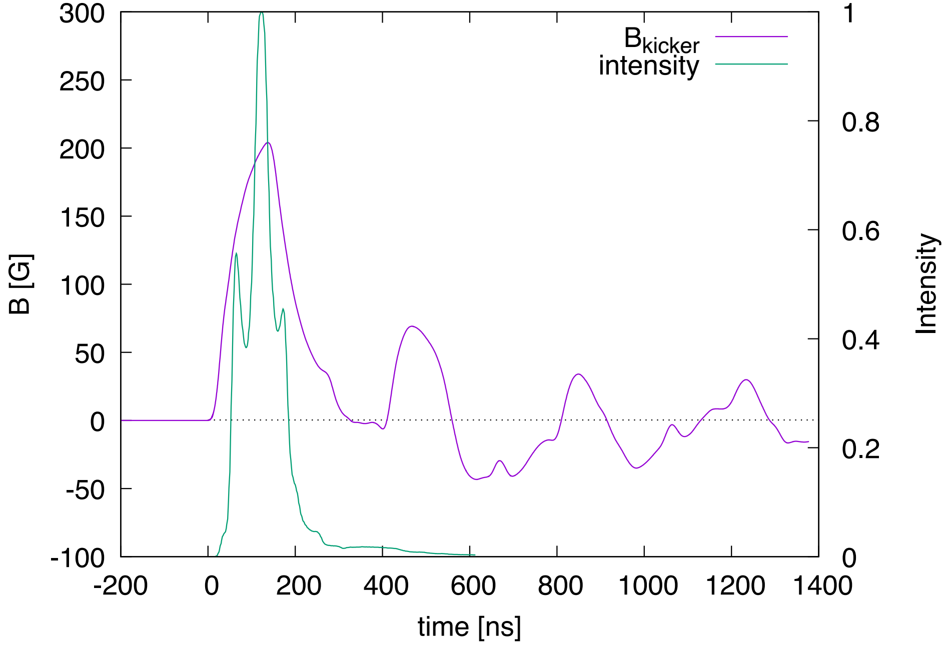
B_{peak} = 204G

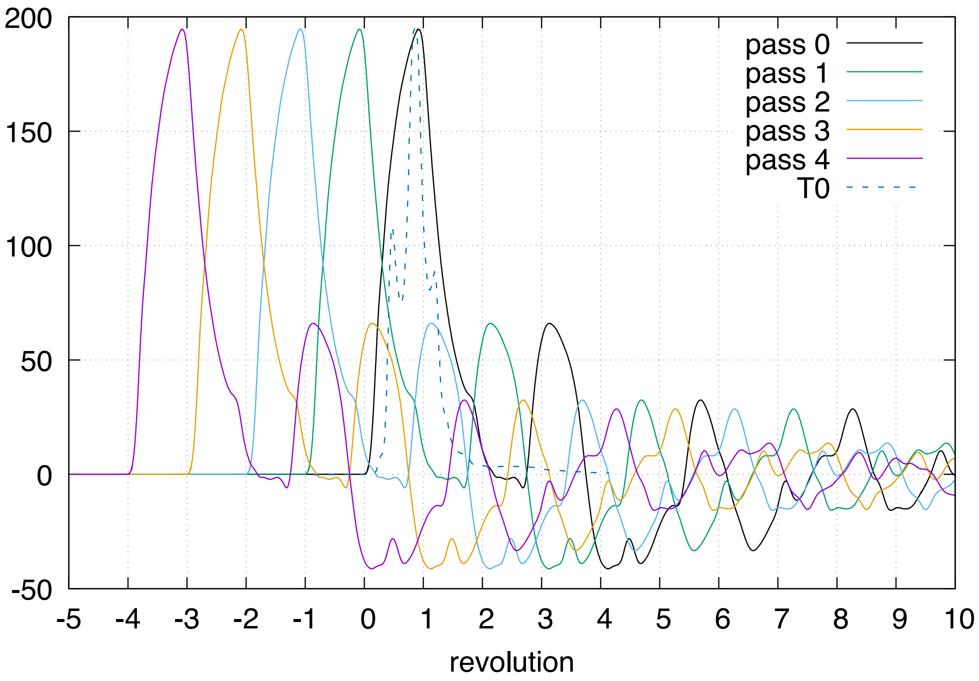
$$x'_{inf} = -2.5 \text{ mrad}$$

Angular distribution of injected particles from simulation



Multiple kicks



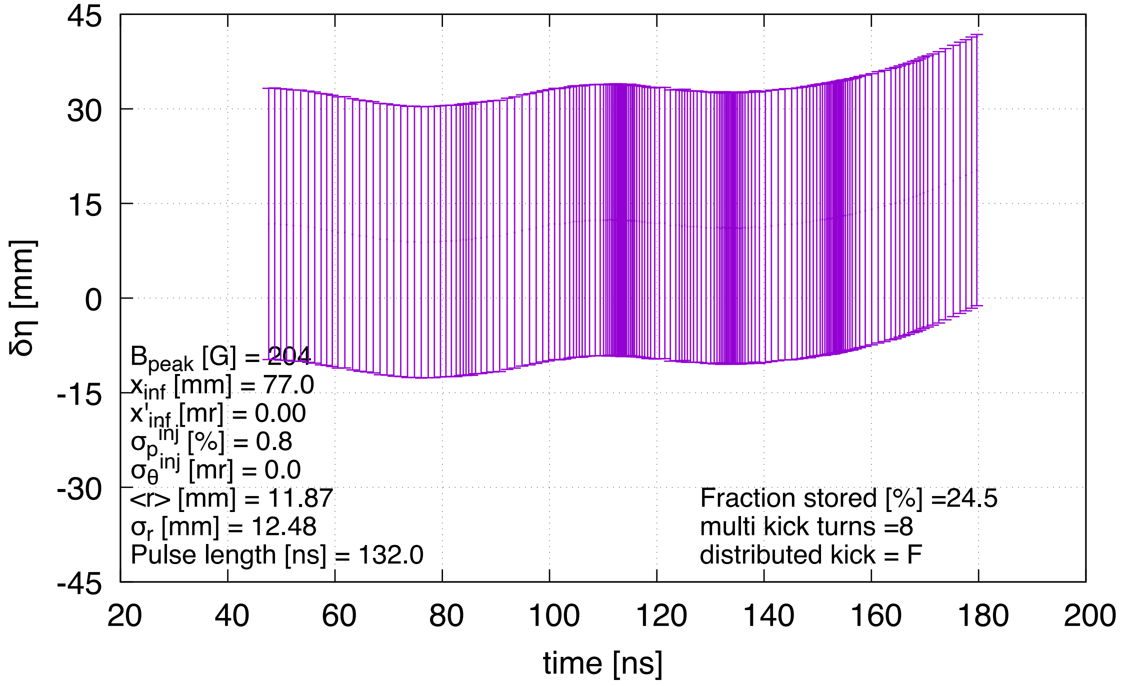


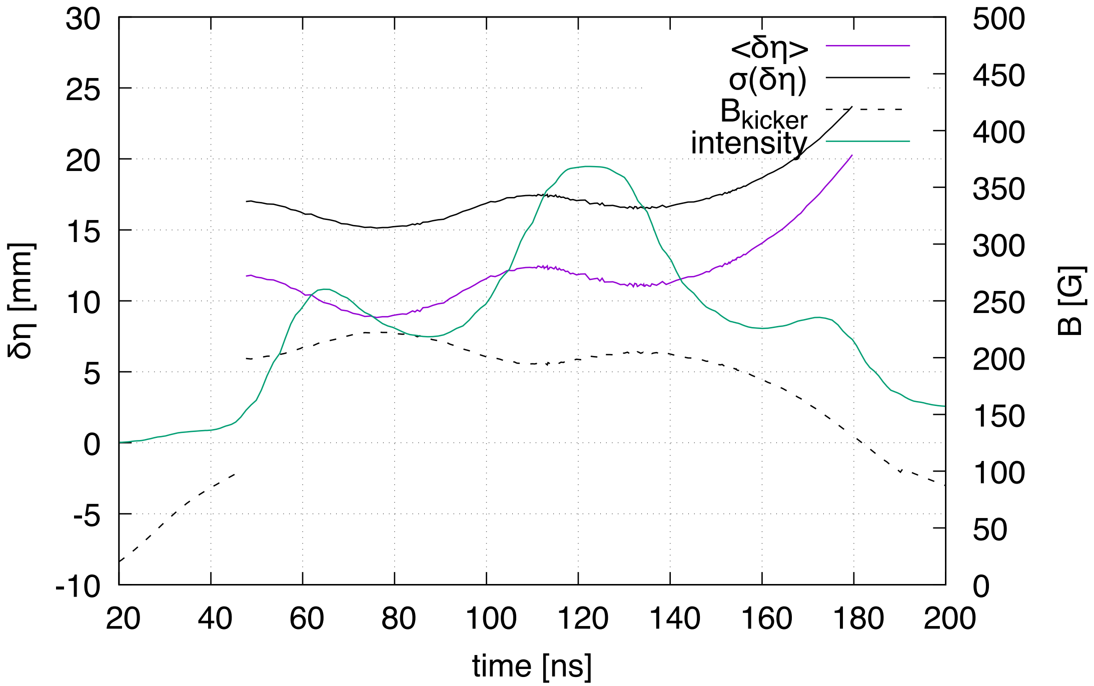
Inequality that defines momentum bite is generalized to include kicks on multiple turns with the replacement

$$\overline{x'_{inf}} = x'_{inf} + \sum_n \theta_k^n \cos(2\pi\nu[\frac{1}{4} + n])$$

$$\overline{\theta_k} = \sum_n \theta_k^n \sin(2\pi\nu[\frac{1}{4} + n])$$

Extended kicker pulse alters momentum time correlation





Field index = 0

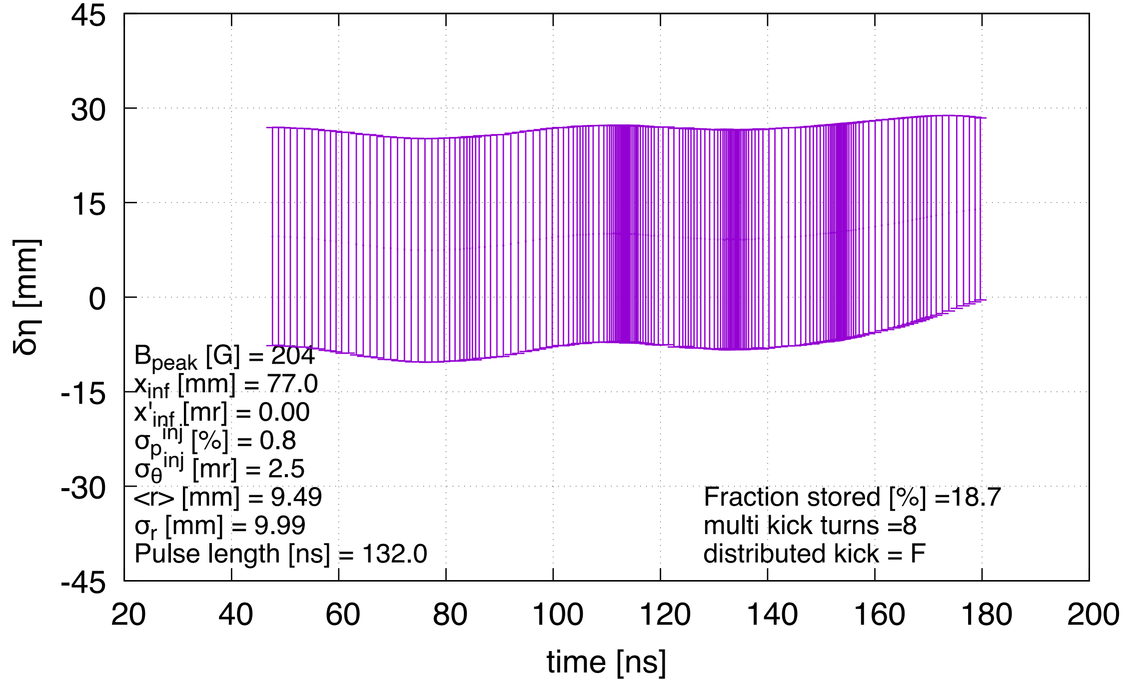
B_{peak} =204 G

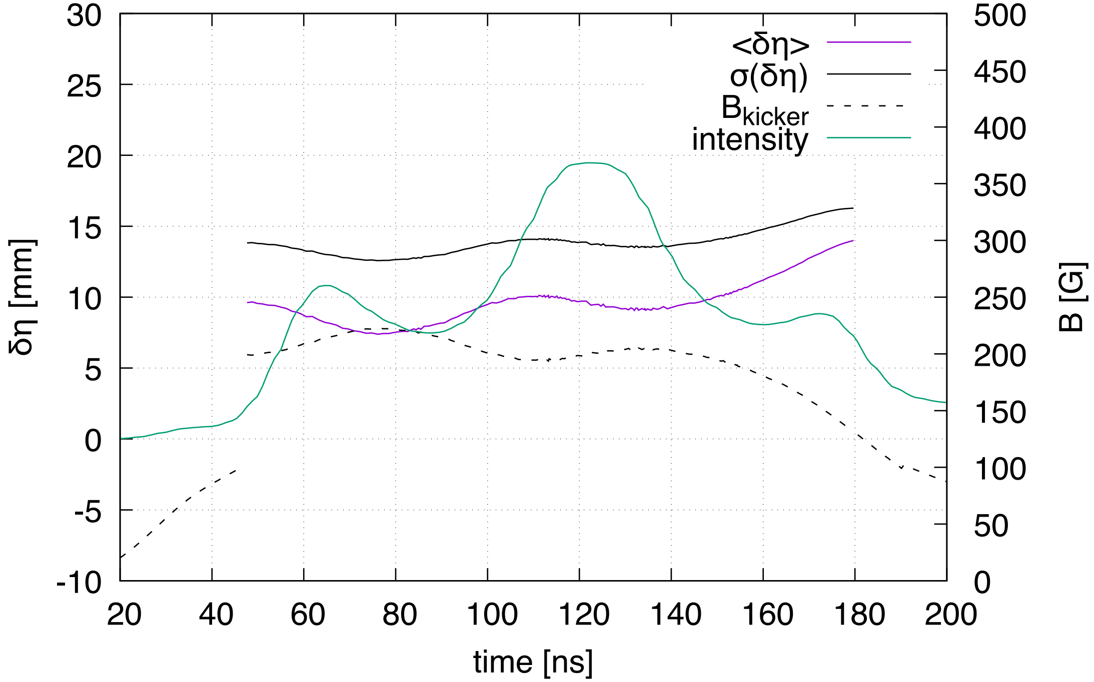
Sum kicks on first 9 passes through kicker

Assumes no collimation for first 9 turns

$$\phi_k = \pi/2, \quad x'_{inf} = 0$$

Multiple kicks and finite injection angle





Field index = 0

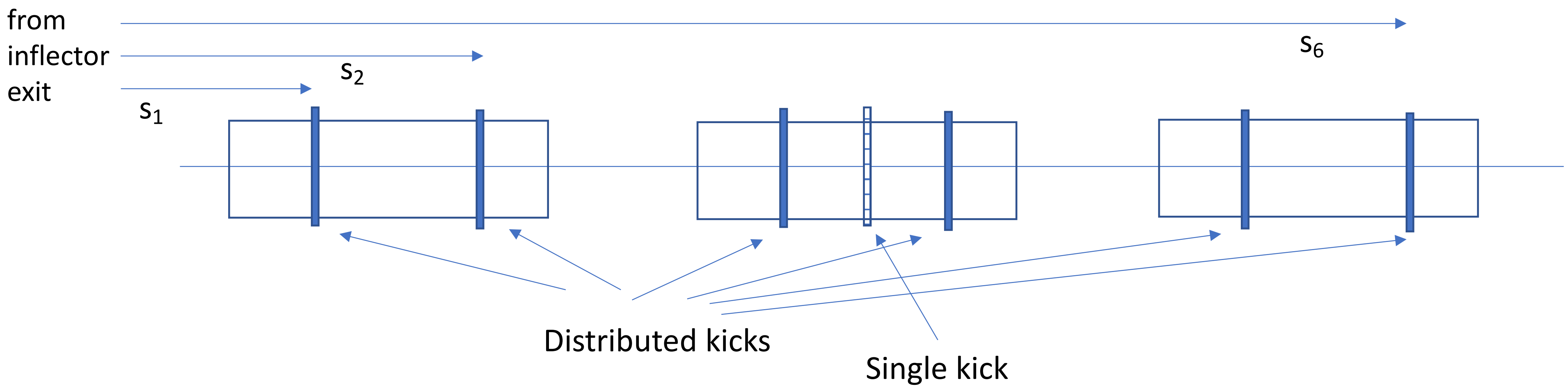
B_{peak} =204 G

Sum kicks on first 9 passes through kicker

$$x'_{inf} = -2.5 \text{ mrad}$$

Distributed kick

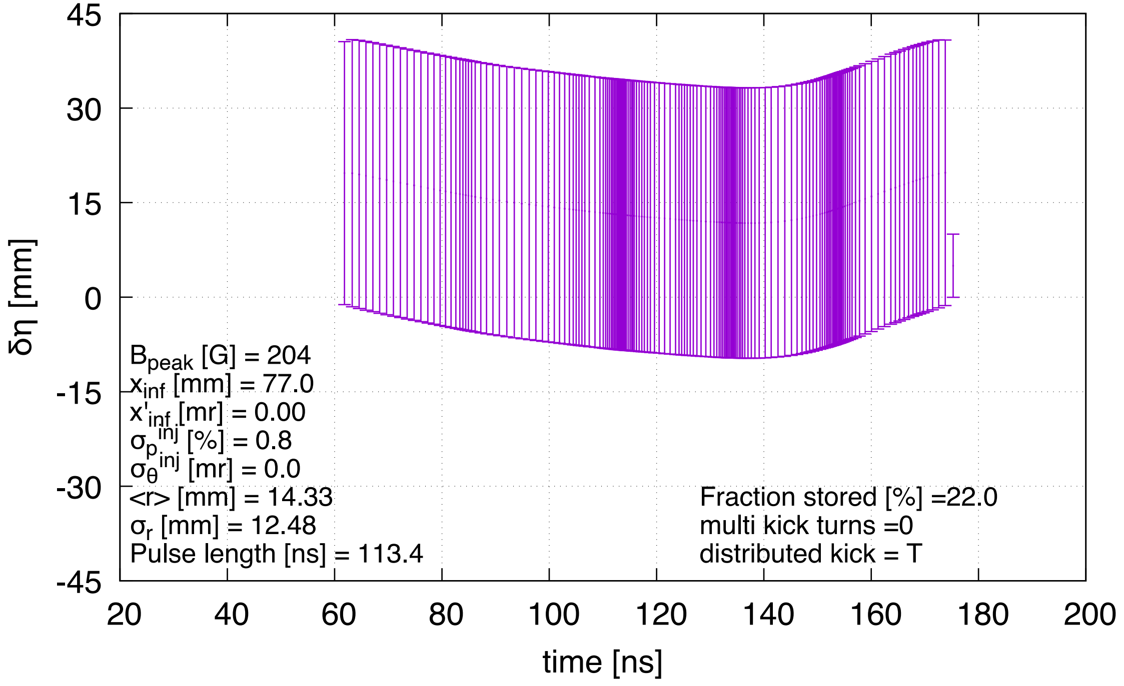
The kicker extends over nearly 5 meters.
Model the extended kick by replacing the single kick with 6 kicks each with 1/6 strength and located in the middle of the first and second half of each kicker

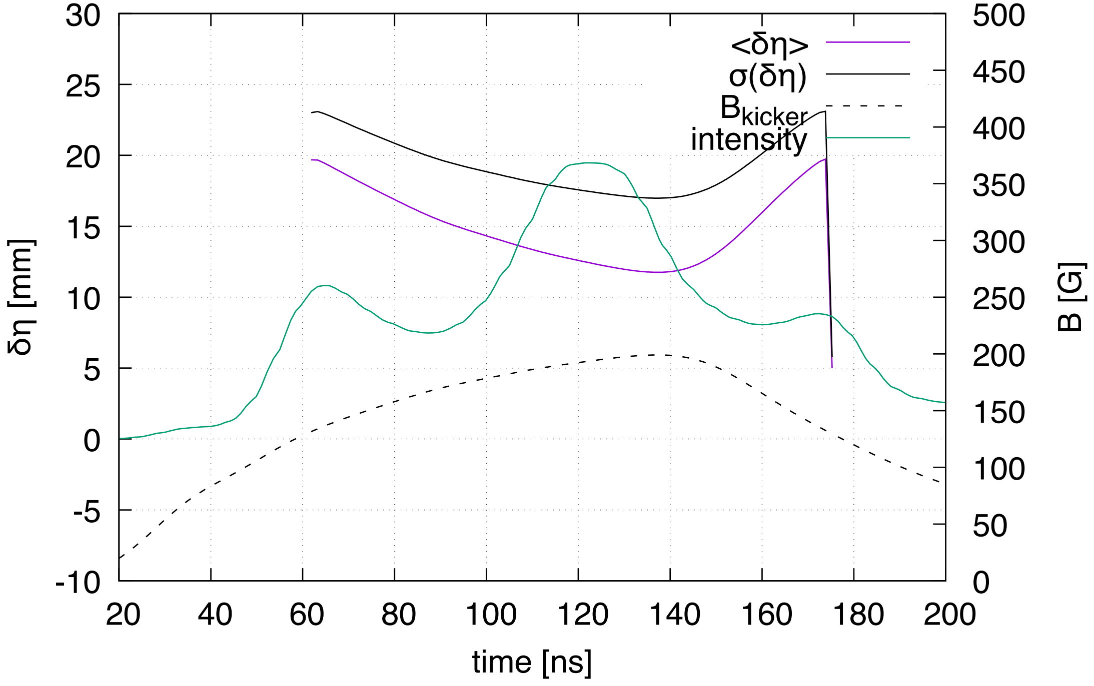


$$\Delta x'_{inf} = \sum_n^N \sum_i^6 \frac{\theta_k^n}{6} \cos(2\pi\nu[\frac{s_i}{R} + n])$$

$$\overline{x'_{inf}} = x'_{inf} + \Delta x'_{inf}$$

$$\overline{\theta_k} = \sum_n^N \sum_i^6 \frac{\theta_k^n}{6} \sin(2\pi\nu[\frac{s_i}{R} + n])$$





Field index = 0

B_{peak} =204 G

$$x'_{inf} = 0$$

$i = 1 \ldots 6$ kicks at

$$\phi_k^i = 2\pi Q_x \frac{s_i}{R_0}$$

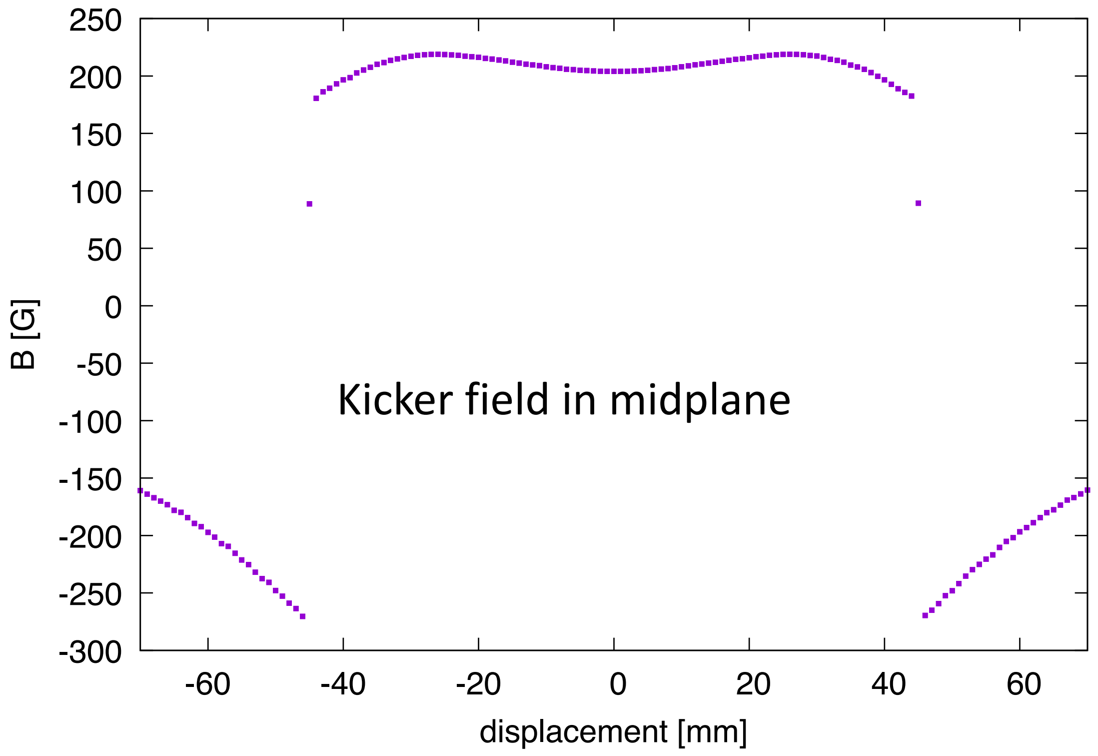
where s_i are the locations of the kicks

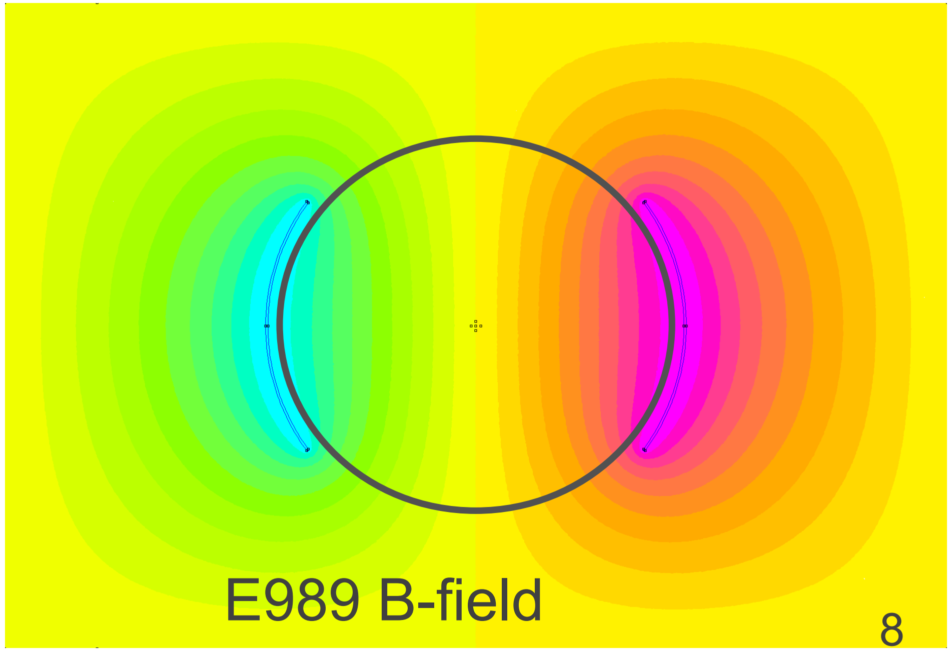
Approximations and limitations

The model assumes

- Continuous quads
- Uniform kicker field and *infinite aperture*
- Continuous collimation
  - => Beginning after the last kick in the sum

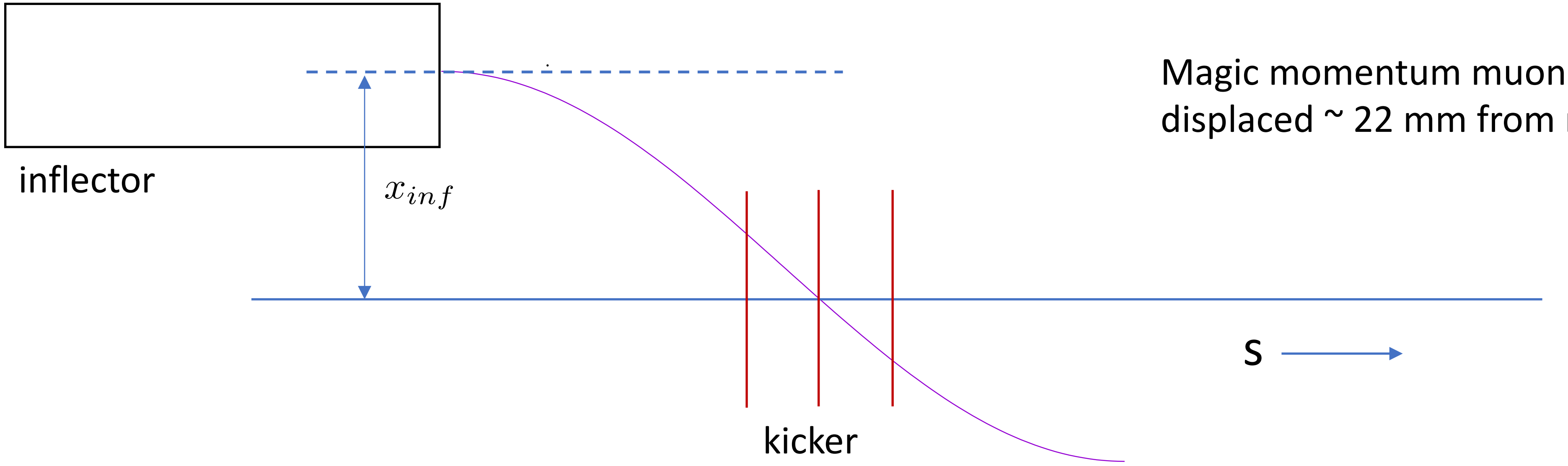
Kicker field is not uniform across the aperture



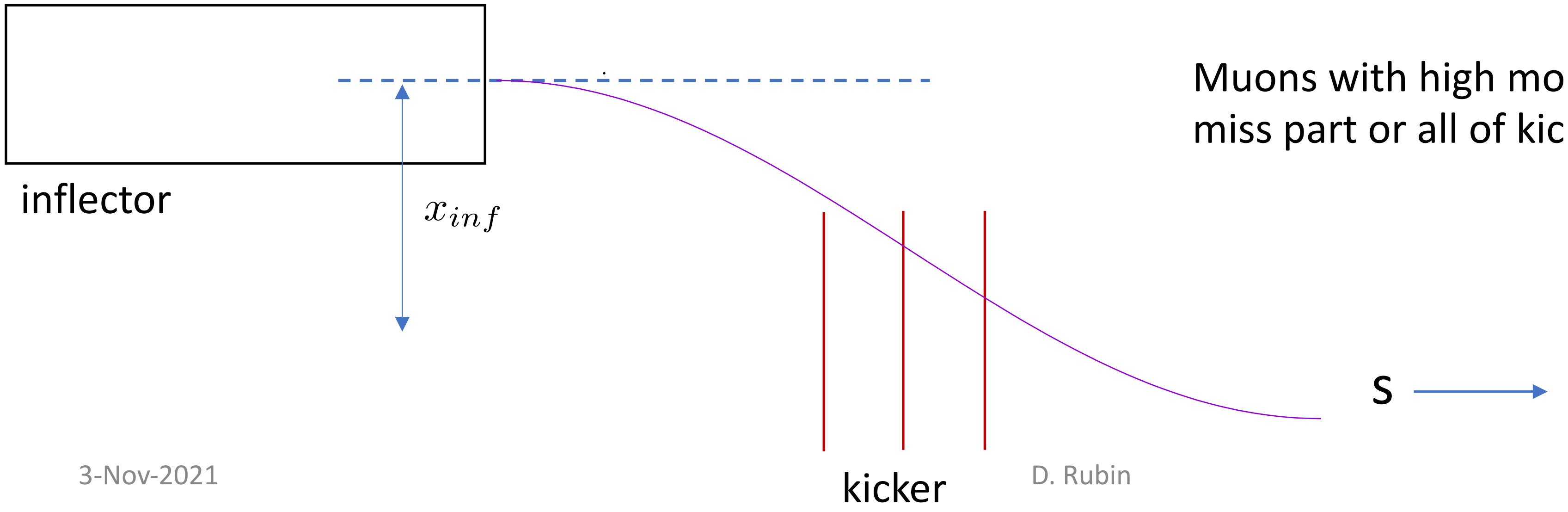


whereas the model assumes the kick is independent of displacement

The kickers have a finite aperture



Magic momentum muon enters kicker #1 displaced ~ 22 mm from magnet axis



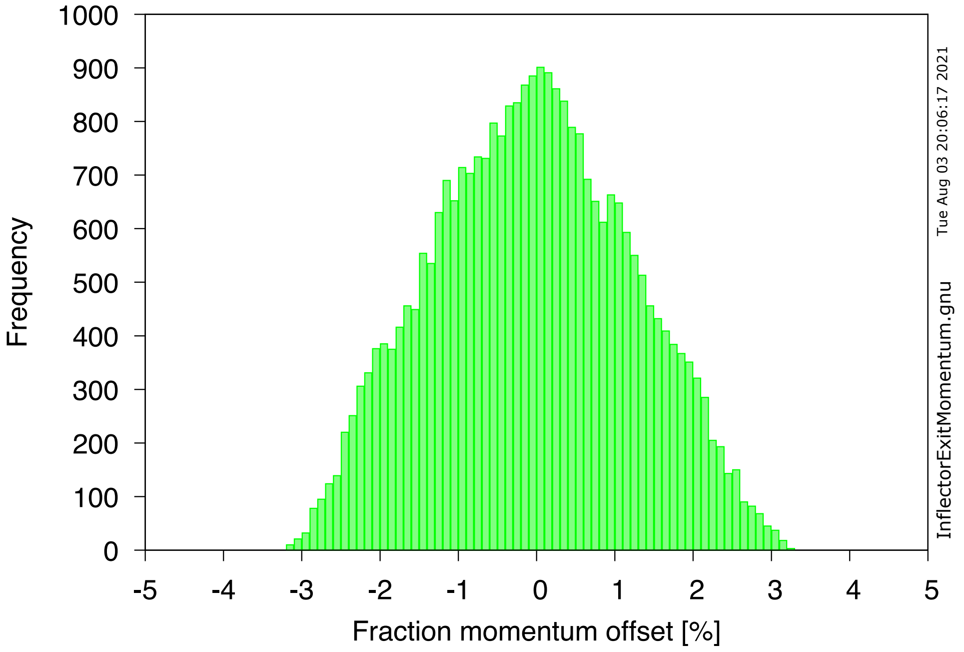
Muons with high momentum and/or angle will miss part or all of kicker #1.

Kicker aperture

Because of the finite kicker aperture,
the model will overestimate acceptance of high momentum muons.

Conclusions:

- Width of accepted momentum bite is nearly independent of kick (as long as the kick is above threshold)
- Centroid of the accepted momentum bite decreases with increasing kick
- Accepted momentum bite narrows with increasing injection angle
- The kicker pulse extends over multiple turns. The effect of the multiple kicks is to ‘flatten’ the pulse and reduce the dependence of momentum on time. So it is important to get the details right in simulations
- Finite kicker aperture reduces acceptance of high momentum particles.
- Details of collimation are important as they determine impact of secondary ... kicks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700
600
500
400
300
200
100
0
-5
-4
-3
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-1
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1
2
3
4
5
Fraction momentum offset [%]
Tue Aug 03 20:06:17 2021
InflectorExitMomentum.gnu


σ=0.8%
Frequency
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200
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100
50
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0
0.1
0.2
0.3
0.4
0.5
Fraction momentum offset [%]